STOKES THEOREM

REMINDEERS. The \textbf{curl} of a vector field \( F \) is

\[
\text{curl}(F) = \nabla \times F = (Q_y - Q_z, P_z - P_x, Q_x - P_y) .
\]

The flux integral of a vector field \( F \) through a surface \( S = \mathcal{R}(R) \) is defined as

\[
\int_{\mathcal{R}} F \cdot n \, ds .
\]

The line integral of a vector field \( F \) along a curve \( C = \mathcal{R}([a, b]) \) is given as

\[
\int_{\mathcal{R}} F \cdot dr = \int_{a}^{b} F(r(t)) \cdot r'(t) \, dt .
\]

The picture shows a tornado near Cordell, Oklahoma. Date: May 22, 1981. Photo Credit: NOAA Photo Library, NOAA Central Library.

The boundary \( C \) of a vector field \( F \) through a surface \( S \) on the surface. By the Maxwell equation, this is

\[
\oint_{\mathcal{R}} \text{curl}(F) \cdot dS = \int_{\mathcal{R}} F \cdot dr .
\]

Note: the orientation of \( C \) is such that if you walk along the surface (head into the direction of the normal \( n \times r )\), then the surface to your left.

EXAMPLE. Let \( F(x, y, z) = (−y, x, 0) \) through the surface parameterized by \( r(u, v) = (u \cos(v), u \sin(v), v) \) on \( R = [0, 2\pi] \times [0, \pi/2] \) and \( r_u \times r_v = \sin(v)\mathbf{e}_u \times \mathbf{e}_v \), so that \( \text{curl}(F) = (0, 0, 1) \). The integral \( \int_{0}^{2\pi} \int_{0}^{\pi/2} \sin(2v) \, dv \, du = 2\pi \).

The boundary \( C \) is parameterized by \( r(t) = (\cos(t), \sin(t), 0) \) so that \( dr = r'(t) \, dt = (−\sin(t), \cos(t), 0) \, dt \) and \( F(r(t)) \cdot r'(t) \, dt = \sin(t)^2 + \cos(t)^2 = 1 \). The line integral \( \int_{C} F \cdot dr \) along the boundary is \( 2\pi \).

SPECIAL CASE: GREEN'S THEOREM. If \( S \) is a surface in the \( x - y \) plane and \( F = \mathcal{R}(P, Q, 0) \) has zero \( z \) component, then \( \text{curl}(F) = (0, 0, P_z - P_y) \) and \( \text{curl}(F) \cdot dS = (Q_x - P_y) \, dx \, dy \).

STOKES THEOREM. Let \( S \) be a surface with boundary curve \( C \) and let \( F \) be a vector field. Then

\[
\int_{C} \text{curl}(F) \cdot dS = \int_{\mathcal{R}} F \cdot dr .
\]

Note: the orientation of \( C \) is such that if you walk along the surface (head into the direction of the normal \( r_u \times r_v \), then the surface to your left.

EXAMPLE. Calculate the flux of the curl of \( F(x, y, z) = (−y, x, 0) \) through the surface parameterized by \( r(u, v) = (u \cos(v), u \sin(v), v) \). Because the surface has the same boundary as the upper hemisphere, the integral is again \( 2\pi \).

For every surface bounded by \( C \) the flux of \( \text{curl}(F) \) through the surface is the same. The flux of the curl of a vector field through a surface \( S \) depends only on the boundary of \( S \).

For every curve between two points \( A, B \) the line integral of \( \text{grad}(f) \) along \( C \) depends only on the end points of \( C \).

BIOT-SA V ARD LAW. A magnetic field \( B \) in absence of an electric field satisfies a Maxwell equation \( \text{curl}(\mathcal{E}) = \frac{4\pi}{c} \mathcal{J} \), where \( \mathcal{J} \) is the total current passing through the wire. The magnetic field satisfies \( B = \frac{2\mathcal{J}}{c} \mathbf{r} \). This is called the Biot-Savard law.

THE DYNAMO. FARADAY'S LAW. The electric field \( E \) and the magnetic field \( B \) are linked by a Maxwell equation \( \text{curl}(\mathcal{E}) = \frac{4\pi}{c} \mathcal{J} \cdot \mathbf{r} \). Take a closed wire \( C \) which bounds a surface \( S \) and consider \( \oint_{\mathcal{R}} B \cdot dS \), the flux of the magnetic field through \( S \). Its change can be related to a voltage using Stokes theorem: \( \frac{d}{dt} \oint_{\mathcal{R}} B \cdot dS = \oint_{\mathcal{R}} \frac{\partial}{\partial t} \mathcal{E} \cdot dS = -\oint_{\mathcal{R}} E \mathcal{J} \mathbf{r} \cdot dS \), where \( U \) is the voltage measured at the cut-up wire. It means that if we change the flux of the magnetic field through the wire, then this induces a voltage. The flux can be changed by changing the amount of the magnetic field but also by changing the direction. If we turn around a magnet around the wire, we get an electric voltage. This happens in a power-generator like an alternator in a car. In practical implementations, the wire is turned inside a fixed magnet.