The arc length is independent of the parameterization of the curve.

\[ r'(t) = (x'(t), y'(t)) \] position
\[ r'(t) = (x'(t), y'(t)) \] velocity
\[ |r'(t)| = \sqrt{x'(t)^2 + y'(t)^2} \] speed
\[ r''(t) = (x''(t), y''(t)) \] acceleration
\[ \kappa = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} \] curvature

**Example:** The circle parameterized by \( r(t) = (r \cos(t), r \sin(t)) \) has the velocity \( v(t) = r \omega = r (-\sin(t), \cos(t)) \) and the unit tangent vector \( \hat{T}(t) = \frac{v(t)}{|v(t)|} \). The arc length parameter is \( s(t) = \int_0^t |v(s)| \, ds \). Because \( |v(t)| = r \sqrt{\omega^2} \), we have

\[ s(t) = \frac{r \omega^2}{2} t^2 \]

**Remark:** Often, there is no closed formula for the arc length of a curve. For example, the Lissajous figure \( r(t) = (\cos(3t), \sin(5t)) \) has the length \( \int_0^{\pi/2} \sqrt{9 \sin^2(3t) + 25 \cos^2(5t)} \, dt \). This integral must be evaluated numerically. If you do the Mathematica Lab, you will see how to do that with the computer.

\[ \kappa = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} \] curvature

**Example:** The circle.

\[ r(t) = (r \cos(t), r \sin(t)) \]
\[ r'(t) = (-r \sin(t), r \cos(t)) \]
\[ |r'(t)| = r \]
\[ T(t) = (-\sin(t), \cos(t)) \]
\[ T'(t) = (-\cos(t), -\sin(t)) \]
\[ \kappa(t) = |T(t)/|r'(t)| = 1/r. \]

**Interpretation.**

If \( s = \int_0^t |r'(t)| \, dt \), then \( s'(t) = ds/dt = |r'(t)| \). Because \( T'(t) = dT/ds = T'(t)/|r'(t)| = \kappa(t) \).

The curvature is the length of the acceleration vector if \( r(t) \) traces the curve with constant speed 1.

A large curvature at a point means that the curve is strongly bent. Unlike the acceleration or the velocity, the curvature does not depend on the parameterization of the curve. You "see" the curvature, while you "feel" the acceleration.

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