Extended hour to hour syllabus

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Maths 21a, Summer 2006

1. Week: Geometry and Space

27. June: Space, coordinates, distance

Class starts with a short slide show highlighting some points of the syllabus and the material which waits for you. Then we dive right into the subject. The idea to use coordinates to describe space was promoted by René Descartes in the 16th century at about the time, when Harvard College was founded. A fundamental notion is the distance between two points. Pythagoras' theorem allows to measure a concrete distance in some Bostonian unit. In order to get a feel about space, we look at some geometric objects defined by coordinates. We will focus on circles and spheres and learn how to find the midpoint and radius of a sphere given as a quadratic expression in x, y, z. This method is called completion of the square. We will discuss, what distinguishes Euclidian distance from other distances. An other more philosophical question is why our physical space is three-dimensional. A further topic for discussion is the existence of other coordinate systems like the photographers coordinate system. Finally, we might mention GPS as an application of distance measurement or the open problem to find a perfect cube, a cube which the length of all sides, side diagonals as well as space diagonals are integers and which will be a homework problem ...

28. June: Vectors, dot product, projections

Two points P, Q define a vector \( \vec{PQ} \). This includes the case \( P = Q \), where \( \vec{PQ} \) is the null vector. The vector connects the initial point \( P \) with the end point \( Q \). Vectors can be attached everywhere in space but are identified if they have the same length and direction. Vectors can describe for example velocities, forces or color or data. We learn first algebraic operations of vectors like addition, subtraction and scaling. This is done both graphically as well as algebraically. We introduce then the dot product between two vectors which results in a scalar. Using the dot product, we can compute length, angles and projections. By assuming the trigonometric cos-formula, we prove the important formula \( \vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\|\cos(\alpha) \), which relates length and angle with the dot product. This formula has some consequences like the Cauchy-Schwartz inequality or the Pythagoras theorem. We mention the notation \( i, j, k \) for the unit vectors.

29. June: Cross product, lines

The third and last lecture of the first week deals with the cross product of two vectors in space. The result of this product in a new vector perpendicular to both. The product can be used for many things. It is useful for example to compute areas, it can be used to compute the distance between a point and a line. It will also be important for constructions like to get a plane through three points or to find the line which is in the intersection of two planes. The cross product is introduced as a determinant. We will prove the important formula \( \|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\|\sin(\alpha) \) and interpret it geometrically as the parallelepiped spanned by \( \vec{v} \) and \( \vec{w} \). In general, there are different ways to describe a geometric object. For lines, we will see a parametric description, as well as an implicit description. The later symmetric equation will later be identified as the intersection of two planes. The simplest equations are linear equations. A linear equation \( ax + by + cz = c \) in three variables geometrically defines a plane. This equation can be written as \( a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \) where \( (x_0, y_0, z_0) \) is a point on the plane. The equation can be interpreted as the place which is perpendicular to the vector \( \vec{u} = (a, b, c) \). We will then learn how to visualize a plane using traces and intercepts. A basic construction is to find the equation of a plane which passes through three points \( P, Q, R \). As an application, we look at some distance formulas like the distance from a point to a plane, the distance from a point to a line as well as the distance between two lines. These distance formulas are geometrically useful. They illustrate how one can use the dot and cross product to measure in space.

2. Week: Functions and Surfaces

4. July: Functions, graphs, quadratics

As the name “multi-variable calculus” suggests, functions of several variables play an essential role in this course. In multivariable calculus, the focus is on functions of two or three variables. The \( \mathbf{P} \), \( \mathbf{Q} \), and \( \mathbf{R} \) we look at some distance formulas as well as the distance between two lines. These distance formulas are geometrically useful. They illustrate how one can use the dot and cross product to measure in space.

5. July: Implicit and parametric surfaces

Surfaces can be described in two fundamental ways: implicitly or parametrically. The implicit description is \( g(x, y, z) = 0 \) like the sphere \( x^2 + y^2 + z^2 - 1 = 0 \), the parametric description is \( \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)) \). In many cases, it is possible to go from one form to the other. There are four important types of surfaces for which one can do that: spheres, ellipsoids, cones, cylinders, paraboloids and hyperboloids. You will have to know the names of these animals in the zoo of functions.

6. July: Implicit and parametric surfaces

28. June: Vectors, dot product, projections

Curves are one-dimensional objects. One can look at curves both in the plane as well as in space, they can take many different shapes. A special case are closed curves in space which are called knots. We will learn how to describe curves by parametrization \( \mathbf{r}(t) = (x(t), y(t), z(t)) \). By differentiation, one obtains the velocity \( \mathbf{r}'(t) \) and the acceleration \( \mathbf{r}''(t) \), which are both vectors. The speed of a curve at some point is the length \( |\mathbf{r}'(t)| \) of the velocity vector. The usual one-dimensional chain rule \( \frac{d}{dt} \left( f(t(s)) \right) = f'(t(s)) \mathbf{r}'(s) \) gives the velocity after a change of time.

3. Week: Curves and Partial Derivatives

11. July: Curves, velocity, acceleration, chain rule

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We first derive the arc length formula for of a curve. This length can be computed by evaluating a one-dimensional integral, by integrating up the symmetric equation \( \frac{1}{2} \mathbf{r}'(t(s)) \mathbf{r}'(s) \) gives the velocity after a change of time.
Triple Integrals and Line Integrals

13. July: First midterm (on week 1-2)

4. Week: Extrema and Lagrange Multipliers

18. July: Gradient, linearization, tangents, chain rule

The gradient of a function is an important tool to describe the geometry of surfaces. Fundamental is the property that the gradient vector \( \nabla f \) is perpendicular to the implicit surface \( g = c \). This allows us to compute tangent planes and tangent lines as well as to approximate a linear function by a linear function near a point. Many physical laws are actually just linearization of more complicated nonlinear laws.

19. July: Extrema, second derivative test

A central application of multi-variable calculus is to extremize functions of two variables. One first identifies critical points, points where the gradient vanishes. The nature of these critical points can be established using the second derivative test. There will be three fundamentally different cases: local maxima, local minima as well as saddle points.

20. July: Extrema with constraints

The topic with maybe the most applications both in science or economics is to extremize a function \( f(x,y) \) in the presence of a constraint \( g(x,y) = 0 \). A necessary condition for a critical point is that the gradients of \( f \) and \( g \) are parallel. This leads to equations called the Lagrange equation.

5. Week: Double Integrals and Surface Integrals

25. July: Double integrals, type I,II regions

Integration in two dimensions is first done on rectangles, then on regions bound by graphs of functions. Similar than in one dimension, there is a Riemann sum approximation of the integral. This allows us to prove results like Fubinis theorem on the change of the integration order. An application of double integration is the computation of area.

26. July: Polar coordinates, surface area

Many regions can be described better in polar coordinates. Examples are so called roses which trace flower-like shapes in the plane but are graphs in polar coordinates. Changing coordinates comes with an integration factor which can be explained also after introducing the surface area.

27. July: Second midterm (on week 3-4)

1. August: Triple integrals, cylindrical coordinates

Triple integrals allow the computation of volumes, moment of inertias or centers of masses of solids. First introduced for cubes it is then extended to more general regions bound by graphs of functions of two variables. Some regions can be described better in cylindrical coordinates, the analogue of polar coordinates in space.

2. August: Spherical coordinates, vector fields

Spherical coordinates allow an even more elegant computation of triple integrals for certain regions like cones or spheres. Next, we will introduce vector fields. They occur as force fields or velocity fields or mechanics and are closely related to the field of ordinary differential equations. Vector fields will occupy us until the end of the course.

3. August: Line integrals, fundamental thm of line integrals

Line integrals are one dimensional integrals along a curve in the presence of a vector field. If the vector field is a force field, then the line integral has the interpretation work done, when walking along the path. For a class of vector fields which we call conservative vector fields one can compute the line integral easily using an identity called the fundamental theorem of line integrals.

8. August: Curl and Green theorem

Greens theorem relates a line integral along a closed curve with a double integral of a derivative of the vector field in the region enclosed by the curve. The theorem is useful for example to compute areas. It also allows an easy computation of line integrals in certain cases. We will see a derivative of the vector field which is called the "curl". It is a scalar field which measures the vorticity of the vector field in the plane.

9. August: Curl and Stokes theorem

Stokes theorem is Greens theorem lifted into three dimensions, where the region is replaced by a surface. Again, one can replace the line integral along the boundary of the surface by an integral of the "curl" of the field over the surface. This integral is a flux integral. The curl of a vector field in three dimensions is a vector field itself. The three components give the vorticity of the vector field in the x,y and z direction.

10. August: Div and Gauss theorem

Finally, the divergence of a vector field inside a solid is related to the flux of the vector field through the boundary of the surface using the divergence theorem which is sometimes also called Gauss theorem. The divergence theorem relates the "local expansion rate" of a vector field with the flux through a closed surface and is useful for example to compute the gravitational field inside a solid.

15. August: Final exam (on week 1-7) 1:30 PM