GREEN’S THEOREM

Maths21a, O. Knill

LINE INTEGRALS (RECALL). If $F(x,y) = (P(x,y), Q(x,y))$ is a vector field and $C : r(t) = (x(t), y(t)), t \in [a,b]$ is a curve, then

$$\int_C F \cdot ds = \int_a^b F(r(t)) \cdot (r'(t), y'(t)) \, dt$$

is called the line integral of $F$ along $C$. It’s helpful to think of the integral as the work of a force field $F$ along $C$. It’s positive if “the force is with you”, negative, if you have to “fight against the force”, while going along the path $C$.

THE CURL OF A 2D VECTOR FIELD. The curl of a 2D vector field $F(x,y) = (P(x,y), Q(x,y))$ is defined as the scalar field

$$\text{curl}(F)(x,y) = Q_y(x,y) - P_y(x,y).$$

INTERPRETATION. curl$(F)$ measures the vorticity of the vector field. One can write $\nabla \times F = \text{curl}(F)$ for the curl of $F$ because the 2D cross product of $\langle P_y, P_x \rangle$ with $F = \langle P, Q \rangle$ is $Q_y - P_y$.

EXAMPLES:

1) $F(x,y) = (−y, x)$. curl$(F)(x,y) = 2$.
2) $F(x,y) = \nabla f$. (conservative field = gradient field = potential) Because $P(x,y) = f_x(x,y), Q(x,y) = f_y(x,y)$, we have curl$(F) = Q_x - P_y = f_{xx} - f_{xy} = 0$.

GREEN’S THEOREM. (1827) If $F(x,y) = (P(x,y), Q(x,y))$ is a vector field and $R$ is a region which has as a boundary a piecewise smooth closed curve $C$ traversed in the direction so that the region $R$ is “to the left”. Then

$$\int_C F \cdot ds = \iint_R \text{curl}(F) \, dxdy$$

Note: for a region with holes, the boundary consists of many curves. They are always oriented so that $R$ is to the left.

GEORGE GREEN (1793-1841) was one of the most remarkable physicists of the nineteenth century. He was a self-taught mathematician and miller, whose work has contributed greatly to modern physics.

SPECIAL CASE. If $F$ is a gradient field, then both sides of Green’s theorem are zero:

$$\int_C F \cdot ds = 0$$

$$\iint_R \text{curl}(F) \, dA = 0$$

The fact that curl$(\text{grad}(f)) = 0$ can be checked directly but it can also be seen from $\nabla \times \nabla f$ and the fact that the cross product of two identical vectors is 0. One just has to treat $\nabla$ as a vector.

APPLICATION: CALCULATING LINE INTEGRALS. Sometimes, the calculation of line integrals is harder than calculating a double integral. Example: calculate the line integral of $F(x,y) = (x^2 - y^2, 2xy)$ along the boundary of the rectangle $[0,2] \times [0,1]$. Solution: curl$(F) = Q_x - P_y = 2y - 2y = 0$ so that

$$\int_C F \cdot ds = \int_0^1 \int_0^2 4y \, dy \, dx = 2y^2|_0^1 = 2 = 4.$$

Remark. One can easily find examples, where the calculation of the line integral is not possible in closed form directly, but where Green allows to do it nevertheless.

APPLICATION: AREA FORMULAS. The vector fields $F(x,y) = (P, Q)$ in the plane have vorticity curl$(F(x,y)) = 0$. The right hand side in Green’s theorem is the area of $R$:

$$\text{Area}(R) = \int_C -y \, dx = \int_C x \, dy$$

EXAMPLE. Let $R$ be the region under the graph of a function $f(x)$ on $[a,b]$. The line integral around the boundary of $R$ is 0 from $f(0)$ to $f(b)$ and $f(a)$ to $f(0)$ because $Q = 0$. The line integral on $(t, f(t))$ is $-\int_0^1 (−y(t), 0) \cdot (1, f'(t))dt = \int_0^1 f(t) \, dt$. Green’s theorem assures that this is the area of the region below the graph.

APPLICATION: THE PLANEIMETER.

The planeimeter is a mechanical device for measuring areas: in medicine to measure the size of the cross-sections of tumors, in biology to measure the area of leaves or wing sizes of insects, in agriculture to measure the area of forests, in engineering to measure the size of profiles... There is a vector field $F$ associated to a planeimeter (put a vector of length 1 orthogonally to the arm). One can prove that $F$ has vorticity 1. The planeimeter calculates the line integral of $F$ along a given curve. Green’s theorem assures it is the area.

The picture to the right shows a Java applet which allows to explore the planeimeter (from a CCP module by O. Knill and D. Winter, 2001). To explore the planeimeter, visit the URL http://www.math.duke.edu/education/ccp/materials/mvcalc/green/