DIVERTON THEOREM

The divergence of a vector field \( F \) is \( \nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \). It is a scalar field.

The flux integral of a vector field \( F \) through a surface \( S \) was defined as \( \iint_S F \cdot dS \). Recall also that the integral of a scalar function \( f \) on a region \( R \) is \( \iiint_R f \, dV = \iiint_R f(x, y, z) \, dx \, dy \, dz \).

GAUSS THEOREM OR DIVERGENCE THEOREM. Let \( G \) be a solid in space. Assume the boundary of the solid is a surface \( S \). Let \( F \) be a vector field. Then

\[
\iiint_G \nabla \cdot F \, dV = \iiint_S F \cdot dS.
\]

The orientation of \( S \) is such that the normal vector \( r_n \times r_u \) points outside of \( G \).

EXAMPLE. Let \( F(x, y, z) = (x, y, z) \) and let \( S \) be sphere. The divergence of \( F \) is constant \( 3 \) and

\[
\iiint_G \nabla \cdot F \, dV = 3 \cdot 4\pi r^3 = 4\pi r^3.
\]

By the divergence theorem, this is \( 4\pi M_3 = 4\pi \iiint_R \rho \, dV \), where \( M_3 \) is the mass of the material inside \( S \).

CONTINUITY EQUATION. If \( \rho \) is the density of a fluid and \( v \) is the velocity field of the fluid, then by conservation of mass, the flux \( \iiint_S \rho \, dV \) of the fluid through a closed surface \( S \) bounding a region \( G \) is \( \int_S \rho \, dV \), the change of mass inside \( G \). But this flux is by Gauss theorem equal to \( \iiint_G \nabla \cdot F \, dV \).

Therefore, \( \iiint_S \rho \, dV = \iiint_G \nabla \cdot F \, dV = 0 \).

WHAT IS THE BOUNDARY OF A BOUNDARY? The fundamental theorem for line integral, Green's theorem, Stokes theorem and Gauss theorem are all of the form \( \int_C \mathbf{F} \cdot d\mathbf{r} = \int_D \nabla \cdot \mathbf{F} \, dV \) where \( dV \) is a derivative of \( F \) and \( dA \) is a boundary of \( A \). They all generalize the fundamental theorem of calculus. There is some similarity in how \( d \) and \( \delta \) behave:

<table>
<thead>
<tr>
<th>( d )</th>
<th>( \delta )</th>
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<tbody>
<tr>
<td>scalar field</td>
<td>( df = \text{curl grad}(f) = 0 )</td>
</tr>
<tr>
<td>surface in space</td>
<td>( dS ) is union of closed curves</td>
</tr>
<tr>
<td>space in space</td>
<td>( dA ) is a closed surface</td>
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The question when \( df(F) = 0 \) implies \( F = \text{curl}(G) \) or whether \( \text{curl}(F) = 0 \) implies \( \text{grad}(G) = 0 \). We look at it Friday.

STOKE'S AND GAUSS. Stokes theorem was discovered 1764 by Joseph Louis Lagrange. Carl Friedrich Gauss, who formulates also Greens theorem, redisCOVERS the divergence theorem in 1813. Gauss also redisCOVERS the divergence theorem in 1825 not knowing of the work of Green and Lagrange.

VOLUME CALCULATION. Similarly as the planimeter allowed to calculate the area of a region by passing along the boundary, the volume of a region can be determined as a flux integral. For example the vector field \( F(x, y, z) = (x, 0, 0) \) which has divergence 1. The flux of this vector field through the boundary of a region is the volume of the region.

\[
\iint_G (x, 0, 0) \, dS = \text{Vol}(G).
\]

GRAVITY INSIDE THE EARTH. How much do we weight deep in earth at radius \( r \) from the center of the earth? (Relevant in the movie "The core")

The law of gravity can be formulated as \( \text{div}(F) = 4\pi \rho \), where \( \rho \) is the mass density. We assume that the earth is a ball of radius \( R \). By rotational symmetry, the gravitational force is normal to the surface: \( F(x) = F(R)x/|x| \).

The flux of \( F \) through a ball of radius \( r \) is \( \iiint_{B_r} F(x) \cdot dV = 4\pi r^2 F(r) \).

By the divergence theorem, this is \( 4\pi M_r = 4\pi \iiint_{B_r} \rho \, dV \), where \( M_r \) is the mass of the material inside \( B_r \).

We have \( (4\pi r)^2 \rho / 3 = 4\pi r^2 F(r) \) for \( r < R \). Inside the earth, the gravitational force \( F(r) = 4\pi r^2 / 3 \). Outside the earth, it satisfies \( F(r) = M/r^2 \) with \( M = 4\pi R^2 \rho / 3 \).

GREEN IDENTITIES. If \( G \) is a region in space bounded by a surface \( S \) and \( f, g \) are scalar functions, then with \( \Delta f = \nabla^2 f = \nabla \cdot \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2} \), one has as a direct consequence of Gauss theorem the first and second Green identities (see homework):

\[
\int_S f \Delta g \, dS = \int_S \nabla f \cdot \nabla g \, dS + \int_S f \, \nabla g \cdot dS.
\]

These identities are useful in electrostatics. Example: if \( g = f \) and \( \Delta f = 0 \) and either \( f = 0 \) on the boundary \( S \) or \( \nabla f \) is orthogonal to \( S \), then Gauss's first identity gives \( \int_S f \nabla^2 f \, dS = 0 \) which means \( f = 0 \) is used to prove uniqueness for the Poisson equation \( \Delta f = 4\pi \rho \) when applying the identity to the difference \( f = \psi_1 - \psi_2 \) of two solutions with different Dirichlet boundary conditions (\( \psi_1 \) on \( S \) or \( \psi_2 \) on \( \psi \) on Neumann boundary conditions (\( \nabla \psi \) orthogonal to \( S \)).

GAUSS THEOREM IN HIGHER DIMENSIONS. If \( G \) is a \( n \)-dimensional "hyperspace" bounded by a \( (n-1) \)-dimensional "hypersurface" \( S \), then \( \iiint_G \nabla \cdot F \, dV = \iiint_S F \cdot dS \).

In dimension \( d \), the divergence is defined \( \text{div}(F) = \nabla \cdot F = \sum_i \partial F_i / \partial x_i \).

The proof of the \( n \)-dimensional divergence theorem is done as in three dimensions.

By the way: Gauss theorem in two dimensions is just a version of Green's theorem. Replacing \( F = (P, Q) \) with \( G = (-Q, P) \) gives \( \text{curl}(F) = \text{div}(G) \) and the flux of \( G \) through a curve is the line integral of \( F \) along the curve. Green's theorem for \( F \) is identical to the 2D-divergence theorem for \( G \).