2D INTEGRALS

1D INTEGRATION IN 100 WORDS. If \( f(x) \) is a continuous function then \( \int_a^b f(x) \, dx \) can be defined as a limit of the Riemann sum \( \sum_{x_k \in [a,b]} f(x_k) \) for \( n \to \infty \) with \( x_k = k/n \). This integral divided by \( |b-a| \) is the average of \( f \) on \([a,b]\). The integral \( \int_a^b f(x) \, dx \) can be interpreted as an area under the graph of \( f \), which can be negative too. If \( f(x) = 1 \), the integral is the length of the interval. The function \( F(x) = \int_a^x f(y) \, dy \) is called an anti-derivative of \( f \). The fundamental theorem of calculus states \( F'(x) = f(x) \). Unlike the derivative, anti-derivatives cannot always be expressed in terms of known functions. An example is: \( f(x) = \cos^2(x) \). The anti-derivative is \( \frac{x}{2} \) – \( \sin(2x)/4 \).

AVERAGES=MEAN. www.worldclimate.com gives the following data for the average monthly rainfall (in mm) for Cambridge, MA, USA (42.38 North 71.11 West.18m Height).

<table>
<thead>
<tr>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
<th>May</th>
<th>Jun</th>
<th>Jul</th>
<th>Aug</th>
<th>Sep</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
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<td>81.4</td>
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<td>67</td>
<td>57.4</td>
<td>61.4</td>
<td>82.8</td>
</tr>
</tbody>
</table>

The average 860.3/12 = 71.7 is a Riemann sum integral.

2D INTEGRATION. If \( f(x,y) \) is a continuous function of two variables on a region \( R \), the integral \( \int_R f(x,y) \, dy \, dx \) can be defined as the limit \( \frac{1}{n^2} \sum_{i,j} f(x_{ij}, y_{ij}) \) for \( n \to \infty \) as \( n \times n \) goes to infinity. If \( f(x,y) = 1 \), then the integral is the area of the region \( R \). The integral divided by the area of \( R \) is the average value of \( f \) on \( R \). For many regions, the integral can be calculated as a double integral \( \int_R \int f(x,y) \, dy \, dx \). In general, the region must be split into pieces, then integrated separately.

One can interpret \( \int_R f(x,y) \, dy \, dx \) as the volume of solid below the graph of \( f \) and above \( R \) in the \( x \)-\( y \) plane. (As in 1D integration, the volume of the solid below the \( x-y \) plane is counted negatively).

EXAMPLE. Calculate \( \int_R f(x,y) \, dy \, dx \), where \( f(x,y) = 4x^2y^3 \) and where \( R \) is the rectangle \([0,1] \times [0,2] \).

\[
\int_0^1 \left( \int_0^2 4x^2y^3 \, dx \right) \, dy = \int_0^2 \left( \int_0^1 x^2(16 - 0) \, dx \right) = 16x^3/3|_0^1 = \frac{16}{3}.
\]

FUBIN’S THEOREM. \( \int_a^b \int_c^d f(x,y) \, dy \, dx = \int_c^d \int_a^b f(x,y) \, dx \, dy \).

TYPES OF REGIONS.

- type I regions
- type II regions

EXAMPLE. Let \( R \) be the triangle \( 1 \geq x \geq 0, 0 \geq y \geq 0, y \leq x \). Calculate \( \int_R e^{-x^2} \, dy \, dx \).

QUANTUM MECHANICS. In quantum mechanics, the motion of a particle (like an electron) in the plane is determined by a function \( u(x,y) \), the wave function. Unlike in classical mechanics, the position of a particle is given in a probabilistic way only. If \( R \) is a region and \( u \) is normalized so that \( \int_R |u(x,y)|^2 \, dx \, dy \leq 1 \), then \( \int_R |u(x,y)|^2 \, dx \, dy \) is the probability that the particle is in \( R \).

EXAMPLE. Unlike a classical particle, a quantum particle in a box \([0, \pi] \times [0, \pi] \) can have a discrete set of energies only. This is the reason for the name “quantum”. If \( -(u_{xx} + u_{yy}) = \lambda u \), then a particle of mass \( m \) has the energy \( E = \frac{\lambda^2}{2m} \). A function \( u(x,y) = \sin(kx) \sin(ny) \) represents a particle of energy \( (k^2 + n^2)\hbar^2/2m \). Let us assume \( \lambda = 2 \) and \( n = 3 \) from now on. Our aim is to find the probability that the particle with energy 18k^2/2m is in the middle 9th cell \( \mathbb{R} \times \mathbb{R} \) of the box.

SOLUTION. We first have to normalize \( u(x,y) = \sin(2x) \sin^2(3y) \), so that the average over the whole square is 1:

\[
A = \int_0^\pi \int_0^\pi \sin^2(2x) \sin^2(3y) \, dx \, dy.
\]

To calculate this integral, we first determine the inner integral

\[
\int_0^\pi \sin^2(2x) \sin^2(3y) \, dx = \frac{1}{8} \sin^2(3y) \left( \frac{1}{2} \sin^2(2x) + \sin^2(4x) \right),
\]

so that the probability amplitude function is \( f(x,y) = \frac{1}{8} \sin^2(2x) \sin^2(3y) \).

The probability that the particle is in \( R \) is slightly smaller than 1/9:

\[
\frac{1}{A} \int_R f(x,y) \, dy \, dx = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{8} \sin^2(2x) \sin^2(3y) \, dx \, dy
\]

\[
= \frac{1}{4\pi^2} \frac{(4x - \sin(4x))/8(2x^3/3)(6x - \sin(6x))/12x^3/3}{1} = 19/9 - 1/(4\sqrt{3})
\]

The probability is slightly smaller than 1/9.

WHERE DO DOUBLE INTEGRALS OCCUR?

- compute areas.
- compute averages. Examples: average rain fall or average population in some area.
- probabilities. Expectation of random variables.
- quantum mechanics: probability of particle being in a region. Find moment of inertia \( \int_R (x^2+y^2)^2 \rho(x,y) \, dx \, dy \).
- find center of mass \( (\int_R x\rho(x,y) \, dx \, dy)/M, \int_R y\rho(x,y) \, dx \, dy)/M \), with \( M = \int_R \rho \, dx \, dy \).
- compute some 1D integrals.

TRIPLE INTEGRALS are defined similarly and covered in detail later. The area under a graph of \( f(x,y) \) can be written as a triple integral. Fubini’s theorem generalizes \( \int f(x,y) \, dx \, dy \) to be: \( \int \int f(x,y,z) \, dx \, dy \, dz \).