**STOKES THEOREM**

**REMINDERS.** The curl of a vector field \( F \) is
\[
\text{curl}(F) = \nabla \times F = (Q_y - P_z, P_x - Q_z, Q_x - P_y).
\]

The flux integral of a vector field \( F \) through a surface \( S = r(R) \) is defined as
\[
\iint_S F \cdot n \, dS = \int_C F \cdot dr.
\]

The line integral of a vector field \( F \) along a curve \( C = r([a, b]) \) is given as
\[
\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) \, dt.
\]

The picture shows a tornado near Cordell, Oklahoma. Date: May 22, 1981. Photo Credit: NOAA Photo Library, NOAA Central Library.

The tornado points into the direction of the field curl, which we align on the \( z \)-axis. To measure the magnetic field at distance \( R > r \) from the wire, we take a curve \( C : r(t) = (\cos(t), \sin(t), 0) \) which bounds a disc. Note that the orientation of \( C \) is such that if you walk along the surface (head into the direction of the normal \( r_u \times r_v \)), then the field points towards you. The flux can be changed by changing the amount of the magnetic field but also by changing the direction.

**EXAMPLE.** Calculate the flux of the curl of \( F(x, y, z) = (-y, x, 0) \) through the surface parameterized by \( r(u, v) = (u \cos(v), u \sin(v), 0) \). The surface has the same boundary when the normal is in the \( -z \)-direction, the integral is again \( 2\pi \) as in the above example.

For every surface bounded by \( C \) the flux of \( \text{curl}(F) \) through the surface is the same. The flux of the curl of a vector field through a surface \( S \) depends only on the boundary of \( S \).

**STOKES THEOREM.** Let \( S \) be a surface with boundary curve \( C \) and let \( F \) be a vector field. Then
\[
\iint_S \text{curl}(F) \cdot dS = \oint_C F \cdot dr.
\]

Note: the orientation of \( C \) is such that if you walk along the surface (head into the direction of the normal \( r_u \times r_v \)), then the surface to your left.

**EXAMPLE.** Let \( F(x, y, z) = (-y, x, 0) \) be the upper semi-hemisphere, then \( \text{curl}(F) = (0, \pi, 0) \) so that the flux of \( \text{curl}(F) \) through the surface is \( 2\pi \).

**SPECIAL CASE: GREEN’S THEOREM.** If \( S \) is a surface in the \( x - y \) plane and \( F = (P, Q, 0) \) has zero \( z \)-component, then
\[
\text{curl}(F) = (0, 0, Q_x - P_y)
\]
and
\[
\iint_S \text{curl}(F) \cdot dS = \iint_S (Q_x - P_y) \, dx \, dy.
\]

**PROOF OF STOKES THEOREM.**

For a surface which is flat, Stokes theorem can be seen with Green’s theorem. If we put the coordinate axis so that the surface is in the \( xy \)-plane, then the vector field \( F \) induces a vector field on the surface such that its 2D curl is the normal component of \( \text{curl}(F) \). The reason is that the third component \( Q_z - P_y \) of \( \text{curl}(F) = (Q_y - P_z, P_x - Q_z, Q_x - P_y) \) is the two dimensional curl: \( \text{curl}(F(x, y, z)) = (0, 0, 1) = Q_z - P_y \). If \( C \) is the boundary of the surface, then
\[
\oint_C F(r(u, v)) \cdot r'(u, v) \, du \, dv = \int_{r_0}^{r_1} \text{curl}(F) \cdot r'(t) \, dt.
\]

For a general surface, we approximate the surface by a mesh of small parallelepipeds. When summing up line integrals along all these parallelepipeds, the line integrals inside the surface cancel and only the integral along the boundary remain. On the other hand, the sum of the fluxes of the curl through boundary adds up to the flux through the surface.

**DISCOVERY OF STOKES THEOREM**

Stokes theorem was found by Ampère in 1825. George Gabriel Stokes (1819-1903) was probably inspired by work of Green and rediscovered the identity around 1840.

George Gabriel Stokes

André Marie Ampère

**BIOT-SAVARD LAW.** A magnetic field \( B \) in absence of an electric field satisfies a Maxwell equation
\[
\text{curl}(E) = 0
\]
and
\[
\text{div}(B) = \frac{J}{\mu_0}
\]
where \( J \) is the current and \( \mu_0 \) is the permeability of free space. To measure the magnetic field at distance \( R > r \) from the wire, we take a closed path \( C \) which bounds a surface \( S \). The line integral of \( B \) along \( C \) is the same. The line integral of the gradient of a function of a curve \( C \) depends only on the end points of \( C \).

For every curve between two points \( A, B \) the line integral of \( \text{grad}(f) \) along \( C \) is the same.

**THE DYNAMO, FARADAY’S LAW.** The electric field \( E \) and the magnetic field \( B \) are linked by a Maxwell equation
\[
\text{curl}(E) = -\frac{d}{dt} B
\]
which relates the change of the magnetic field through a surface \( S \). Its change can be related with a voltage using Stokes theorem:
\[
\oint_C B \cdot ds = \iint_S \text{curl}(E) \cdot dS = \iint_S B \cdot dS = \iint_S \text{curl}(E) \cdot dS
\]
where \( U \) is the voltage measured at the cut-up wire. It means that if we change the flux of the magnetic field through the wire, then this induces a voltage. The flux can be changed by changing the amount of the magnetic field but also by changing the direction.

If we turn around a magnet around the wire, we get an electric voltage. This happens in a power-generator like an alternator in a car. In practical implementations, the wire is turned inside a fixed magnet.