WHY ARE PARTIAL DERIVATIVES IMPORTANT?

- Geometry. For example, the gradient \( \nabla f(x, y, z) \) is a vector normal to a surface at the point \((x, y, z)\). Tangent spaces.
- Approximations, linearizations.
- Partial differential equations: laws describing nature.
- Optimization problems, as we will see later.
- Solution to some integration problems using generalizations of fundamental theorem of calculus.
- In general helpful to understand and analyze functions of several variables.

PARTIAL DERIVATIVES

Partial derivatives are fundamental in understanding and analyzing functions of several variables. For functions with more than one variable, partial derivatives are defined in a similar way to functions of one variable.

\[
\frac{\partial f}{\partial x} (x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}
\]

\[
\frac{\partial f}{\partial y} (x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}
\]

### Example

For the function \( f(x, y) = x^2 - 6xy^2 + y^4 \), the partial derivatives are:

\[
\frac{\partial f}{\partial x} (x, y) = 2x - 6y^2
\]

\[
\frac{\partial f}{\partial y} (x, y) = -12xy + 4y^3
\]

The partial differential equation

\[
\frac{\partial f}{\partial t} (x, t) = \frac{\partial^2 f}{\partial x^2} (x, t) + \frac{\partial^2 f}{\partial x \partial y} (x, t)
\]

is called the Laplace equation, and

\[
\frac{\partial f}{\partial t} (x, t) = \frac{\partial f}{\partial x} (x, t) + \frac{\partial f}{\partial y} (x, t)
\]

is called the advection equation.

**Dirac's Equation**

Paul A. M. Dirac discovered a PDE describing the electron which is consistent both with quantum theory and special relativity.

\[
\begin{align*}
\frac{\partial f}{\partial t} (x, t) & = \frac{\partial^2 f}{\partial x^2} (x, t) + \frac{\partial^2 f}{\partial x \partial y} (x, t) \\
\frac{\partial f}{\partial t} (x, t) & = \frac{\partial f}{\partial x} (x, t) + \frac{\partial f}{\partial y} (x, t) \\
\end{align*}
\]

This won him the Nobel Prize in 1933. Dirac's equation could have two solutions, one for an electron with positive energy, and one for an electron with negative energy. Dirac interpreted the later as an antiparticle: the existence of antiparticles was later confirmed.