LINES and PLANES

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LINES. A point \( P \) and a vector \( \vec{v} \) define a line \( L \). It is the set of points

\[ L = \{ P + t \vec{v}, \text{ where } t \text{ is a real number} \} \]

The line contains the point \( P \) and points into the direction \( \vec{v} \).

EXAMPLE. \( L = \{ (x, y, z) = (1, 1, 2) + t(2, 4, 6) \} \).

This description is called the parametric equation for the line.

EQUATIONS OF LINE. We can write \((x, y, z) = (1, 1, 2) + t(2, 4, 6)\) so that \(x = 1 + 2t, y = 1 + 4t, z = 2 + 6t\). If we solve the first equation for \( t \) and plug it into the other equations, we get \( y = 1 + 2(2x - 3), z = 2 + 3(2x - 2) \).

We can therefore describe the line also as

\[ L = \{(x, y, z) \mid y = 2x - 1, z = 6x - 4 \} \]

SYMMETRIC EQUATION. The line \( \vec{r} = P + t\vec{v} \) with \( P = (x_0, y_0, z_0) \) and \( \vec{v} = (a, b, c) \) satisfies the symmetric equations

\[ \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \]

(every expression is equal to \( t \)).

EXAMPLE. \( L: (x, y, z) = (2, 4, 6) + t(1, 2, 3) \) or \( \vec{v} = (1, 2, 3) \).

DISTANCE POINT-LINE (3D). If \( P \) is a point and \( \vec{v} \) a vector \( L \), then

\[ d(P, L) = \frac{|(P - Q) \cdot \vec{n}|}{|\vec{n}|} \]

is the distance between \( P \) and the line. You recognize that this is just the scalar projection of the vector \( \vec{QP} = P - Q \) onto the vector \( \vec{n} \).

DISTANCE LINE-LINE (3D). If \( P \) is a point in space and \( \vec{v} \) is the line \( \vec{r}(t) = Q + t\vec{u}, M \) is the line \( \vec{r}(t) = P + t\vec{v} \), then

\[ d(L, M) = \frac{|(P - Q) \times \vec{n}|}{|\vec{n}|} \]

is the distance between the two lines \( L \) and \( M \). This formula is verified by writing \((P - Q) \times \vec{n} = |P - Q||\vec{n}||\sin(\theta)\).

DISTANCE POINT-PLANE (3D). If \( P \) is a point in space, then

\[ d(P, L) = \frac{|(P - Q) \cdot \vec{n}|}{|\vec{n}|} \]

is the distance between \( P \) and the plane.

This formula is verified by writing \((P - Q) \times \vec{n} = |P - Q||\vec{n}||\sin(\theta)\).

PLANE THROUGH 3 POINTS \( P, Q, R \). The vector \((a, b, c) = \vec{n} = (Q - P) \times (R - P)\) is normal to the plane. Therefore, the equation is \( ax + by + cz = d \). The constant \( d = ax_0 + by_0 + cz_0 \) because the point \( P = (x_0, y_0, z_0) \) is on the plane.

PLANE THROUGH POINT \( P \) AND LINE \( \vec{r}(t) = Q + t\vec{u}, M \) is the line \( \vec{r}(t) = P + t\vec{v} \), then

\[ d(L, M) = \frac{|(P - Q) \times \vec{n}|}{|\vec{n}|} \]

is the distance between the two lines \( L \) and \( M \). This formula is just the scalar projection of the vector \( \vec{QP} = P - Q \) onto the vector \( \vec{n} \) normal to both \( \vec{v} \) and \( \vec{u} \).

LINES AND PLANES IN MATHEMATICA

Plotting a line.

\[ \text{ParametricPlot3D}[\{1, 1, 1\} + t\{3, 4, 5\}, \{t, -2, 2\}] \]

Plotting a plane.

\[ \text{ParametricPlot3D}[\{1, 1, 1\} + t\{3, 4, 5\} + s\{2, 3, 2\}, \{t, -2, 2\}, \{s, -2, 2\}] \]

Finding the equation of a plane.

\[ P = \{-1, -1, 1\}; Q = \{0, 0, 1\}; R = \{1, 1, 3\}; n = \text{Cross}[Q - P, R - P]; n; \text{n}\{x, y, z\} - n\cdot P \]