LINE INTEGRALS

\[ \int_C F \cdot dr = \int_C F(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \]

is called the line integral of \( F \) along the curve \( C \). If \( F \) is a force, then the line integral is work.

A RIDDLE ABOUT WORK.

1) If you accelerate a car from 0 m/s to 10 m/s = 2.24 miles/hour the kinetic energy needed is mass times \( 10^2 / 2 = 50 \).

2) If you accelerate a car from 10 m/s to 20 m/s = 4.84 miles/hour the kinetic energy has increased by \( 20^2 / 2 - 10^2 / 2 = 200 - 50 = 150 \).

3) Now watch situation 2) from a moving coordinate system which moves with constant 10 m/s. In that system, the car accelerates from 0 to 10 meter per second.

Because all physical laws are the same in different coordinate systems (going from one to the other is a Galileo transformation), the energy should be the same when accelerating from 0 to 10 or from 10 to 20.

This is in contradiction to the fact that accelerating the car from 0 to 10 needs three times less energy than accelerating the car from 10 to 20. Why?

EXAMPLE. Compute the line integral for the vector field \( F(x, y) = (x^2, y^2) \) along the boundary of a square in the coordinate direction.

We split up the path into four paths:
\( \gamma_1 : r(t) = (t, 0) \) for \( t \in [0, 1] \),
\( \gamma_2 : r(t) = (1 - t, 0) \) for \( t \in [0, 1] \),
\( \gamma_3 : r(t) = (0, t) \) for \( t \in [0, 1] \),
\( \gamma_4 : r(t) = (0, 1 - t) \) for \( t \in [0, 1] \),
then \( \gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \).

The integral is then
\[ \int_{C} F \cdot dr = \int_{\gamma_1} F \cdot dr + \int_{\gamma_2} F \cdot dr + \int_{\gamma_3} F \cdot dr + \int_{\gamma_4} F \cdot dr \]

Plugging in \( F \cdot dr = F(r(t)) \cdot r'(t) \) gives
\[ \int_{0}^{1} 2t^2 \cdot 1 \, dt + \int_{0}^{1} (1 - 2t)^2 \cdot 1 \, dt + \int_{0}^{1} (1 - t)^2 \cdot 1 \, dt + \int_{0}^{1} 2 \, dt = 0. \]

A PERPETUUM MOTION MACHINE. Before entering high school, one of my passions was to construct "perpetuum motion machines". Physics classes in school quickly killed that dream. But still, these machines still fascinate me, especially if they work! Will will see more about these machines next week, but here is a model which allows us to practice line integrals.

Problem: by arranging cleverly charged plates (see lecture), we realize a static electric field \( \vec{E} = (0, 0, x) \). We construct a wire along the elliptical path \( C : r(t) = (\cos(t), 0, 3\sin(t)) \). What is the Voltage
\[ V = \int_{C} \vec{E}(r(t)) \cdot r'(t) \, dt \]

we measure at the wire?

Answer: \( \int_{0}^{2\pi} 3 \cos^2(t) \, dt = 3\pi \).

30 second background info in electricity: when an electron is moved around in an electric field along a path \( C \), it gains the potential \( \int_{C} \vec{E} \cdot dr \). This potential is also called "voltage" and measured in "Volts". When moving a charge through a voltage difference, it gains some energy. This energy is proportional to the amount of charge going through the wire as well as the voltage \( V \). If the charge is 1F which is "current” times “time”, then the energy is \( V \cdot \text{IT} \), which can also be seen as the power \( V \cdot \text{I} = \text{Watts} \) multiplied with time. For example, through a light bulb of 100W, a current of about one amperre flows if the voltage is 110 Volts. If we let the lamp burn for 10 hours, we use the energy 10 \cdot 100Wh which is one Watt hour. In Massachusetts, a Kilowatt hour costs about 10 – 15 cents.

Running a 100 Watt light bulb for 24 hours costs a quarter.