SOLUTION TO NASH’S PROBLEM

Nash’s problem given with the encouraging words

"It might solve it in a few months. Most of you however will need a life-time”.

to a multivariable calculus class in the movie ”A beautiful mind” is:

NASH’s PROBLEM. Find a subset $X$ of $\mathbb{R}^3$ with the property that $V/W$ is 8-dimensional, where $V$ is the set of vector fields $F$ on $\mathbb{R}^3 \setminus X$ which satisfy $\text{curl}(F) = 0$ and where $W$ is the set of vector fields $F$ which are conservative $F = \nabla f$.

The SOLUTION OF NASH’s PROBLEM:

Let $X$ be the union of 8 distinct parallel lines $X = \bigcup_{i=1}^{8} \{(x_i, y_i, z) \mid -\infty < z < \infty \}$ in $\mathbb{R}^3$.

The vector fields $F_i(x, y, z) = (-y - y_i)/((x-x_i)^2 + (y-y_i)^2), (x-x_i)/((x-x_i)^2 + (y-y_i)^2), 0)$.

Every vector field which satisfies $\text{curl}(F) = 0$ outside $X$.

Every vector field which satisfies $\text{curl}(F) = 0$ in $\mathbb{R}^3 \setminus X$ can be written as

$$F = G + \sum_{i=1}^{8} a_i F_i,$$

where $a_i$ are some real numbers and where $G = \nabla g$ is a vector field which is a gradient.

BACKGROUND. Even so the problem can be posed and solved in a multi-variable course (as in this summer school), the problem rather belongs to a more advanced algebraic topology course, to be fully appreciated. Nash’s problem is the inverse cohomology problem to find a manifold $M$ with a 8-dimensional fundamental group.

DE RHAM’S THEOREM. A special case of ”de Rham theorem” states that on a manifold $M$, the vector space of all 1-forms $F$ satisfying $dF = 0$ modulo the space of all 1-forms $F$ which are of the form $F = dG$ is the same as the first cohomology group $H^1(M)$, which is equal to the fundamental group of $M$.

The dimension of the fundamental group of $M$ is the maximal number of closed paths, which you can find in $M$ so that no path can be deformed inside $M$ to any other other path in that family.

In 3 dimension, 1-forms can be associated with vector fields. For every 1-form, $dF$ is a 2-form which is the curl of $F$. In 3 dimensions, 2-forms can be identified with vector fields. If $G$ is a 0-form, a smooth function on $M$, then $dG$ is the gradient of $G$.

To find a space $X$ in Nash’s problem, one has to find a manifold with a 8-dimensional fundamental group. Taking away 8 lines from three dimensional space is one of the possibilities. A closed path which winds around one of the lines and no other line can not be deformed to a point (otherwise, people could steel your bike chained to a pole), nor can it be deformed to a path which winds around an other line.

The solution with lines is not unique. One could take for example the union of 8 closed arbitrary unlinked and unknotted curves in space. It would be hard however to come up in general with explicit vectorfields $F_i$, whose existence is assured by de Rham’s theorem.