Solutions to Problem Set 3

July 23, 2002

Part 1

1.a.

\[ r(t) = (t^3, t^2) \]
\[ r'(t) = (3t^2, 2t) \]
\[ r''(t) = (6t, 2) \]

The curve looks like a parabola but with the opposite concavity so it has a cusp at the origin.

b.

\[ r(t) = (\sin(t), t^3, \cos(t)) \]
\[ r'(t) = (\cos(t), 3t^2, -\sin(t)) \]
\[ T(t) = \frac{1}{\sqrt{1 + 9t^4}}(\cos(t), 3t^2, -\sin(t)) \]

The curve looks like a spiral in the direction of the y axis that has been compressed near the origin and stretched out at infinity.

2.

\[ r''(t) = (\cos(t), -\cos(3t), .2t) \]
\[ r'(t) = (\sin(t), -\frac{1}{3}\sin(3t), .1t^2) \]
\[ r(t) - r(0) = (-\cos(t), \frac{1}{9}\cos(3t), \frac{1}{30}t^3) \]
3. \( r(t) = (t^2, 1 + t, 1 + t^3) \). \( r(-1) = (1, 0, 0) \), so \((1, 0, 0)\) lies on the curve. As for the point \((1, 2, -2)\), \( x = 1 \Rightarrow t^2 = 1 \) thus \( t = 1 \) or \(-1\). \( y = 2 \Rightarrow 1 + t = 2 \) thus \( t = 1 \). But, \( z = -2 \Rightarrow 1 + t^3 = -2 \) thus \( t = (3)^{\frac{1}{3}} \) or \(-3)^{\frac{1}{3}}\). This yields a contradiction, so \((1, 2, -2)\) does not lie on the curve.

\[
\begin{align*}
    r'(t) &= (2t, 1, 3t^2) \\
    r''(t) &= (2, 0, 6t)
\end{align*}
\]

so \( r'(-1) = (-2, 1, 3) \) \( r''(-1) = (2, 0, -6) \) (4)

4.

\[
\begin{align*}
x^2 + y^2 &= t^2(\cos(t))^2 + t^2(\sin(t))^2 \\
&= t^2((\cos(t))^2 + (\sin(t))^2) \\
&= t^2 \cdot 1 \\
&= t^2 = z
\end{align*}
\]

Thus, the curve is a spiral painted on the paraboloid \( x^2 + y^2 = z \).

5. \( x(t) = 2 \cos(t) \) and \( y(t) = 2 \sin(t) \) so \( z(t) = 4 \cos(t) \sin(t) = 2 \sin(2t) \).

\( r'(t) = (-2 \sin(t), 2 \cos(t), 4 \cos(2t)) \).

Part 2

1. \( r'(t) = (2t, t \sin(t), t \cos(t)) \). So, \( |r'(t)| = \sqrt{4t^2 + t^2} = \sqrt{5}t \) for \( t > 0 \).

\[
\int_0^\pi \sqrt{5}t \, dt = \frac{\sqrt{5}}{2} \pi^2
\]

2. \( r'(t) = (2t, 2, \frac{1}{t}) \). So, \( |r'(t)| = \sqrt{4t^2 + 4 + (\frac{1}{t})^2} = \sqrt{(2t + \frac{1}{t})^2} = 2t + \frac{1}{t} \) for \( t > 0 \).

\[
\int_1^e 2t + \frac{1}{t} \, dt = t^2 + \ln(t)|_1^e = e^2 - 1 + 1 - 0 = e^2
\]

Note that in the original problem the lower limit was 0. For this case the arc length is \( \infty \).
3. 

\[ r'(t) = (e^t \cos(t) - e^t \sin(t), e^t \cos(t) + e^t \sin(t), 1) \]
\[ r''(t) = (2e^t \sin(t), 2e^t \cos(t), 0) \]

so \( r'(0) = (1, 1, 1) \) and \( r''(0) = (0, 2, 0) \)

(6)

\[ |r'(0) \times r''(0)| = \left| \begin{array}{ccc} i & j & k \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{array} \right| = |(-2, 0, 2)| = 2\sqrt{2} \]

Thus, \( K(0) = \frac{\sqrt{24}}{9} \).

4. 

\[ x = u \]
\[ y = v \]
\[ z = -\sqrt{-2u^2 - 4v^2 + 1} \]

(7)

One could also use the parametrization,

\[ x = \sqrt{\frac{1}{2} \cos(u) \sin(v)} \]
\[ y = \frac{1}{2} \sin(u) \sin(v) \]
\[ z = \cos(v) \]

(8)

where \( 0 \leq u < 2\pi \) and \( \frac{\pi}{2} \leq v \leq \pi \).

5. 

\( (x, y, z) = ((2 + \cos(u)) \cos(v), (2 + \cos(u)) \sin(v), \sin(u)) \)

where \( 0 \leq u < 2\pi \) and \( 0 \leq v < 2\pi \).