Solutions to Problem Set 1

July 9, 2002

Part 1

1. \(x + 2y = 4\) is a plane.

2. a. The distance between \(P = (3, 7, 5)\) and the z-axis is the distance between \(P\) and the closest point on the z-axis, which is \((0, 0, 5)\). So,

\[
\text{dist}(P, \text{z-axis}) = \sqrt{3^2 + 7^2} = \sqrt{58}
\]

b. This is just the absolute value of the z-coordinate. ans = 5.

c. \(\sqrt{3^2 + 7^2 + 5^2}\)

3. Consider the equation for the sphere with radius \(\sqrt{7}\) and center \((6, 5, -2)\).

\[(x - 6)^2 + (y - 5)^2 + (z + 2)^2 = 7\]

The trace in xy plane is achieved by setting \(z = 0\) in this equation. Similarly, the xz and yz traces are achieved by setting \(y = 0\) and \(x = 0\), respectively.

So for the xy trace consider,

\[
(x - 6)^2 + (y - 5)^2 + 2^2 = 7
\]

\[
(x - 6)^2 + (y - 5)^2 = 3.
\]

(1)

Thus, the xy trace is a circle with center \((6, 5, 0)\) and radius \(\sqrt{3}\).

For the xz trace consider,

\[
(x - 6)^2 + 5^2 + (z + 2)^2 = 7
\]

\[
(x - 6)^2 + (z + 2)^2 = -18.
\]
The left side of the equation is positive, whereas the right side is negative. Thus, there is a contradiction and the xz trace is empty. A similar calculation shows that the yz trace is empty.

4.

\[
4x^2 + 4y^2 + 4z^2 - 8x + 16y = 1
\]

\[
4(x^2 - 2x) + 4(y^2 + 4y) + 4z^2 = 1
\]

\[
4(x^2 - 2x + 1) + 4(y^2 + 4y + 4) + 4z^2 = 1 + 4 + 16
\]

\[
4(x - 1)^2 + 4(y + 2) + 4z^2 = 21
\]

\[
(x - 1)^2 + (y + 2)^2 + z^2 = \frac{21}{4}
\]

So this is the equation for a sphere of radius \( \frac{\sqrt{21}}{2} \) and center \((1, -2, 0)\).

5. \(4 < x^2 + y^2 + z^2 < 9\). The inequalities are strict because we want the points that are between but not on the spheres.

Part 2

1. \(a + b = (-1, 2) + (5, 3) = (4, 5). \quad a - b = (-6, -1).\)

2. \(|v| = \sqrt{2^2 + 4^2 + 2^2} = \sqrt{24}. \) So the vector with length 6 and the same direction as \(v\) will be \(6 \frac{v}{|v|} = (-\sqrt{6}, 2\sqrt{6}, \sqrt{6}).\)

3. Two vectors are orthogonal when their dot product is zero. Thus, \(v \cdot w = (-6, b, 2) \cdot (b, b^2, b) = -6b + b^3 + 2b = b^3 - 4b = 0.\) Factoring the equation we see, \(b(b - 2)(b + 2) = 0.\) \(b = 0\) gives the zero vector. Whether the zero vector is orthogonal to \(v\) or whether the word orthogonal connotes non-zero vectors is a semantic argument and I did not take points off for including or not including \(b = 0\) as an answer. Otherwise, \(b = 2, -2.\)

4. The diagonal of the unit cube is \(a = (1, 1, 1)\) and \(b = (1, 1, 0)\) is a diagonal of one of its faces. If \(\theta\) is the angle between the two diagonals then \(a \cdot b = 2 = |a||b|\cos(\theta) = \sqrt{3}\sqrt{2}\cos(\theta).\) So, \(\theta = \arccos\left(\frac{2}{\sqrt{6}}\right) = \arccos\left(\frac{\sqrt{3}}{\sqrt{2}}\right).\)

5. \(\text{proj}_a b = a(a \cdot b)/|a|^2 = -\frac{4}{25}(3, 0, 4)\)

and

\(\text{comp}_a b = |\text{proj}_a b| = \frac{4}{5}.\)
Given that the book gives a slightly different definition of the scalar projection, I also accepted \( \text{comp}_a b = -\frac{4}{5} \).

Part 3

1. 

\[
(-3, 2, 2) \times (6, 3, 1) = \begin{vmatrix} i & j & k \\ -3 & 2 & 2 \\ 6 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 2 & -3 \\ 3 & 1 & 6 \\ 6 & 3 & 1 \end{vmatrix} k = -4i + 15j - 21k.
\]

The vector pointing in this direction and of unit length is \( \frac{1}{\sqrt{682}} (-4, 15, -21) \).

2. First we calculate the normal to the plane. The normal will be perpendicular to \((2, -4, 6) - (1, 1, 1) = (1, -5, 4)\) and \((5, 1, 3) - (1, 1, 1) = (4, 0, 2)\) so \( n = (1, -5, 4) \times (4, 0, 2) = (-5(2) + 4(0))i - (1(2) - 4(4))j + (1(0) - (-5)4)k = -10i + 14j + 20k \). Thus the equation for the plane is \(-10(x - 1) + 14(y - 1) + 20(z - 1) = 0\) or \(-5x + 7y + 10z = 14\).

3. The equations for the planes are

\[
\begin{align*}
x + y + z &= 1 \\
x + z &= 0
\end{align*}
\]

Subtracting the second equation from the first we see that \( y = 1 \). That combined with second equation, \( x = -z \), gives you the parametric form for the line by letting \( x = t \) we get

\[
\begin{align*}
x &= t \\
y &= 1 \\
z &= -t
\end{align*}
\]

4. Let \( L \) be the given line. Then \((1, 1, 0)\) is the point on \( L \) corresponding to \( t = 0 \). \( L \) is in the direction of \( a = (1, -1, 2) \) and \( b = (1, 0, 2) \) is the vector joining \((1, 1, 0)\) and \((0, 1, 2)\). Then

\[
b - \text{proj}_a b = (-1, 0, 2) - \frac{(1, -1, 2) \cdot (-1, 0, 2)}{1^2 + (-1)^2 + 2^2} (-1, -1, 2) = (-\frac{3}{2}, \frac{1}{2}, 1)
\]
is a direction vector for the required line. So one parametric form for the line is

\[
\begin{align*}
x &= -\frac{3}{2}t \\
y &= 1 + \frac{1}{2}t \\
z &= 2 + t
\end{align*}
\]

5. By equation 8 in the book on page 682, the distance is

\[
D = \frac{1}{\sqrt{16 + 36 + 1}} \left[ 4(3) + (-6)(-2) + 1(7) - 5 \right] = \frac{26}{\sqrt{53}}
\]