Lagrange Multipliers

The method of Lagrange multipliers allows us to maximize or minimize functions with the constraint that we only consider points on a certain surface. To find critical points of a function $f(x, y, z)$ on a level surface $g(x, y, z) = C$ (or *subject to the constraint* $g(x, y, z) = C$), we must solve the following system of simultaneous equations:

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$
$$g(x, y, z) = C$$

Remembering that ∇f and ∇g are vectors, we can write this as a collection of four equations in the four unknowns x, y, z, and λ:

$$f_x(x, y, z) = \lambda g_x(x, y, z)$$
$$f_y(x, y, z) = \lambda g_y(x, y, z)$$
$$f_z(x, y, z) = \lambda g_z(x, y, z)$$
$$g(x, y, z) = C$$

The variable λ is a dummy variable called a "Lagrange multiplier"; we only really care about the values of x, y, and z.

Once you have found all the critical points, you plug them into f to see where the maxima and minima. The critical points where f is greatest are maxima and the critical points where f is smallest are minima.

Solving the system of equations can be hard! Here are some tricks that may help:

1. Since we don't actually care what λ is, you can first solve for λ in terms of x, y, and z to remove λ from the equations.
2. Try first solving for one variable in terms of the others.
3. Remember that whenever you take a square root, you must consider both the positive and the negative square roots.
4. Remember that whenever you divide an equation by an expression, you must be sure that the expression is not 0. It may help to split the problem into two cases: first solve the equations assuming that a variable is 0, and then solve the equations assuming that it is not 0.

For problems 1-3,

(a) Use Lagrange multipliers to find all the critical points of f on the given surface (or curve).

(b) Determine the maxima and minima of f on the surface (or curve) by evaluating f at the critical values.

1 The function $f(x, y, z) = x + y + 2z$ on the surface $x^2 + y^2 + z^2 = 3$.

2 The function $f(x, y) = xy$ on the curve $3x^2 + y^2 = 6$.

3 The function $f(x, y, z) = x^2 - y^2$ on the surface $x^2 + 2y^2 + 3z^2 = 1$. (Make sure you find *all* the critical points!)

If the level surface is infinitely large, Lagrange multipliers will not always find maxima and minima.

4 (a) Use Lagrange multipliers to show that $f(x, y, z) = z^2$ has only one critical point on the surface $x^2 + y^2 - z = 0$.

(b) Show that the one critical point is a minimum.

(c) Sketch the surface. Why did Lagrange multipliers not find a maximum of f on the surface?

Lagrange Multipliers – Solutions

1 (a) We have $f(x, y, z) = x + y + 2z$ and $g(x, y, z) = x^2 + y^2 + z^2$, so $\nabla f = \langle 1, 1, 2\rangle$ and $\nabla g = \langle 2x, 2y, 2z\rangle$. The equations to be solved are thus

$$1 = 2\lambda x \tag{1}$$
$$1 = 2\lambda y \tag{2}$$
$$2 = 2\lambda z \tag{3}$$
$$x^2 + y^2 + z^2 = 3 \tag{4}$$

To solve these, note that λ cannot be 0 by the first three equations, so we get

$$x = \frac{1}{2\lambda}, \qquad y = \frac{1}{2\lambda} \qquad \text{and} \qquad z = \frac{1}{\lambda}.$$

Plugging these values into (4) gives

$$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = 3,$$

or $\lambda = \pm\frac{1}{\sqrt{2}}$. Plugging these values of λ into the equations above, the critical points are thus $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\sqrt{2})$.

(b) Since $f(x, y, z) = x + y + 2z$, we have $f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{2}) = 3\sqrt{2}$ and $f(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\sqrt{2}) = -3\sqrt{2}$. Thus $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{2})$ is the maximum and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\sqrt{2})$ is the minimum.

2 (a) We have $f(x, y) = xy$ and $g(x, y, z) = 3x^2 + y^2$, so $\nabla f = \langle y, x\rangle$ and $\nabla g = \langle 6x, 2y\rangle$. The equations to be solved are thus

$$y = 6\lambda x \tag{5}$$
$$x = 2\lambda y \tag{6}$$
$$3x^2 + y^2 = 6 \tag{7}$$

Plugging the first equation into the second gives

$$y = 6\lambda(2\lambda y) = 12\lambda^2 y.$$

If y were 0, then x would be 0 too, which is impossible by (7). Thus we can divide by y to get that $12\lambda^2 = 1$. Now plug the first equations into (7) to get

$$\begin{aligned} 6 &= 3x^2 + (6\lambda x)^2 \\ &= 3x^2 + 36\lambda^2 x^2 \\ &= 3x^2 + 3(12\lambda^2)x^2 \\ &= 3x^2 + 3x^2. \end{aligned}$$

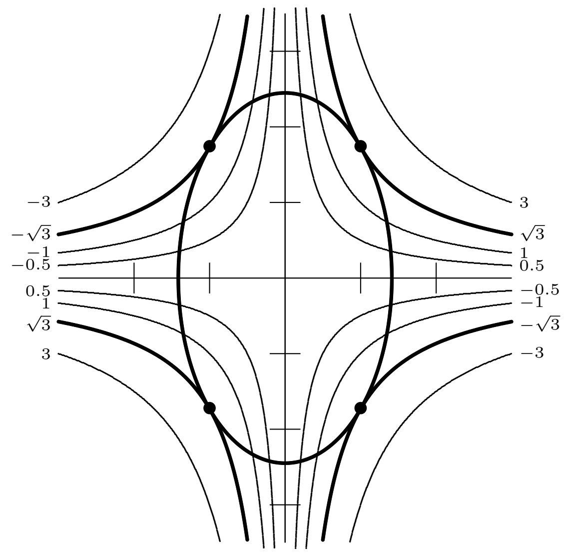
Thus $x = \pm 1$, and $y = \pm\sqrt{3}$ by (7). There are thus four critical points: $(1, \sqrt{3})$, $(1, -\sqrt{3})$, $(-1, \sqrt{3})$, and $(-1, -\sqrt{3})$.

(b) Since $f(x, y) = xy$, we have

$$f(1, \sqrt{3}) = f(-1, -\sqrt{3}) = \sqrt{3}$$
$$f(1, -\sqrt{3}) = f(-1, \sqrt{3}) = -\sqrt{3}.$$

Thus $(1, \sqrt{3})$ and $(-1, -\sqrt{3})$ are maxima and $(1, -\sqrt{3})$ and $(-1, \sqrt{3})$ are minima.

It is instructive to see the picture:



The ellipse is the level curve of $g(x, y)$. The other curves are all various level curves of $f(x, y)$, and the extreme values occur when these level curves share a tangent with the level curve of $g(x, y)$. (These tangent level curves are darker than the other level curves of f.)

3 (a) We have $f(x, y, z) = x^2 - y^2$ and $g(x, y, z) = x^2 + 2y^2 + 3z^2$, so $\nabla f = \langle 2x, -2y, 0\rangle$ and $\nabla g = \langle 2x, 4y, 6z\rangle$. The equations to be solved are thus

$$2x = 2\lambda x \quad (8)$$
$$-2y = 4\lambda y \quad (9)$$
$$0 = 6\lambda z \quad (10)$$
$$x^2 + 2y^2 + 3z^2 = 1 \quad (11)$$

To solve these equations, we look at several cases:

Case 1: $\lambda = 0$

By the first two equations, this implies $x = 0$ and $y = 0$. Thus by (11), $z = \pm\frac{1}{\sqrt{3}}$, and there are two critical points, $(0, 0, \frac{1}{\sqrt{3}})$ and $(0, 0, -\frac{1}{\sqrt{3}})$.

Case 2: $\lambda \neq 0$

By the third equation, this implies $z = 0$.

Case 2a: $x = 0$

Then by (11), $y = \pm\frac{1}{\sqrt{2}}$, and there are two critical points, $(0, \frac{1}{\sqrt{2}}, 0)$ and $(0, -\frac{1}{\sqrt{2}}, 0)$.

Case 2b: $x \neq 0$

By the first equation, this implies $\lambda = 1$. The second equation then becomes $-2y = 4y$, so $y = 0$. Thus by (11), $x = \pm 1$, and there are two critical points, $(1, 0, 0)$ and $(-1, 0, 0)$.

(b) Since $f(x, y, z) = x^2 - y^2$, we have

$$f(0, 0, \tfrac{1}{\sqrt{3}}) = f(0, 0, -\tfrac{1}{\sqrt{3}}) = 0$$
$$f(0, \tfrac{1}{\sqrt{2}}, 0) = f(0, -\tfrac{1}{\sqrt{2}}, 0) = -\tfrac{1}{2}$$
$$f(1, 0, 0) = f(-1, 0, 0) = 1.$$

Thus $(1, 0, 0)$ and $(-1, 0, 0)$ are maxima and $(0, \frac{1}{\sqrt{2}}, 0)$ and $(0, -\frac{1}{\sqrt{2}}, 0)$ are minima. It can be shown that $(0, 0, \frac{1}{\sqrt{3}})$ and $(0, 0, -\frac{1}{\sqrt{3}})$ are saddle points.

4 (a) We have $f(x, y, z) = z^2$ and $g(x, y, z) = x^2 + y^2 - z$, so $\nabla f = \langle 0, 0, 2z\rangle$ and $\nabla g = \langle 2x, 2y, -1\rangle$. The equations to be solved are thus

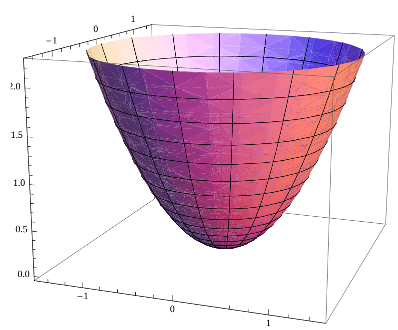
$$0 = 2\lambda x \tag{12}$$
$$0 = 2\lambda y \tag{13}$$
$$2z = -\lambda \tag{14}$$
$$x^2 + y^2 - z = 0 \tag{15}$$

If $\lambda \neq 0$, then $x = 0$ and $y = 0$ by the first two equations, so $z = 0$ by (15). This gives a critical point $(0, 0, 0)$.

If $\lambda = 0$, then $z = 0$ by (14), which implies $x = 0$ and $y = 0$ by (15). Thus we again just get the same critical point $(0, 0, 0)$.

(b) Since $f(x, y, z) = z^2$, $f(x, y, z) \geq 0$ for all (x, y, z). But at our point $(0, 0, 0)$, we have $f(0, 0, 0) = 0$. Thus $(0, 0, 0)$ is a minimum.

(c) This is our standard example of an elliptic paraboloid:



As we can see from the sketch, the surface is infinite, and in particular we can find points (x, y, z) on the surface with z as big as we want. Thus $f(x, y, z) = z^2$ can be as big as we want on the surface, so it has no maximum. That is, the reason Lagrange multipliers did not find a maximum is that there isn't any maximum!