PARTIAL DIFFERENTIAL EQUATIONS

O. Knill, Math 21a

FUNCTIONS OF TWO VARIABLES. We consider functions \( f(x, t) \) in two variables. Viewing the variable \( t \) as time, we can look at the function \( x \mapsto f(x, t) \) of one variable evolving in time. The describing equation is a partial differential equation (PDE). It is a differential equation which involves the derivatives with respect to both space \( x \) and time \( t \). The function \( f(x, t) \) could denote the temperature of a stick or the height of a water wave at position \( x \) and time \( t \).

PARTIAL DERIVATIVES. We write \( f_x(x, t) \) and \( f_t(x, t) \) for the partial derivatives with respect to \( x \) or \( t \). The notation \( f_{xx}(x, t) \) means that we differentiate twice with respect to \( x \).

Example: for \( f(x, t) = \cos(x + 4t^2) \), we have

- \( f_x(x, t) = -\sin(x + 4t^2) \)
- \( f_t(x, t) = -8t \sin(x + 4t^2) \)
- \( f_{xx}(x, t) = -\cos(x + 4t^2) \)

One also uses the notation \( \frac{\partial f}{\partial x} \) for the partial derivative with respect to \( x \). Tired of all the "partial derivative signs", we always write \( f_x(x, y) \) or \( f_t(x, y) \) in this handbook. This is an official abbreviation in the scientific literature.

PARTIAL DIFFERENTIAL EQUATIONS. A partial differential equation is an equation for an unknown function \( f(x, t) \) in which at least two different partial derivatives occur.

- \( f_x(x, t) + f_t(x, t) = 0 \) with \( f(x, 0) = \sin(x) \) has a solution \( f(x, t) = \sin(x - t) \).
- \( f_x(x, t) = f(x, t) \) has the solution \( f(x, 0)e^t \). The equation is not a PDE. Why not?
- \( f_x(x, t) - f_t(x, t) = 0 \) has a solution \( f(x, t) = \sin(x - t) + \sin(x + t) \). Check it!

EXAMPLE: THE WAVE EQUATION. A wave can be modeled by the wave equation

\[ f_{tt}(x, t) + c^2 f_{xx}(x, t) = 0 \]

where \( c \) is a constant, the speed of the waves.

EXAMPLE: THE HEAT EQUATION. The temperature distribution \( f(x, t) \) in a metal wire satisfies the heat equation

\[ f_t(x, t) = \mu f_{xx}(x, t) \]

This PDE tells us the rate of change of the temperature at the point \( x \) is proportional to the second space derivative of \( f(x, t) \) at \( x \). A function \( f(x) = f(0, 0) \) determines an initial temperature distribution. The constant \( \mu \) depends on the heat conductivity of the material. Metals for example conduct heat well and have a large \( \mu \).

VISUALIZATION. We can plot the graph of the function \( f(x, t) \) or the temperature distribution for different times \( t \).

\[ f(x, 0) \quad f(x, 1) \quad f(x, 2) \quad f(x, 3) \quad f(x, 4) \]

EXAMPLE: THE BURGERS EQUATION. The Burgers equation

\[ f_t(x, t) + f(x, t)f_x(x, t) = \mu f_{xx}(x, t) \]

This partial differential equation can have shocks: the waves break. You see that at the beach. With positive \( \mu \), one can give explicit traveling waves \( f(t, x) = (1 + e^{2(x-ct)/4\mu})^{-1} \). Waves \( f(t, x) = \frac{1}{1 + e^{2(x-ct)/4\mu}} \) become discontinuous at \( t = 1 \).

VISUALIZATION. Again we can plot the wave functions \( f(x, t) \) for fixed times \( t \).

\[ f(x, 0) \quad f(x, 25) \quad f(x, 5) \quad f(x, 75) \quad f(x, 0.99) \]

TO THE DERIVATION OF THE HEAT EQUATION. The temperature \( f(x, t) \) is proportional to the kinetic energy at the position \( x \). Divide the stick into \( n \) adjacent cells and assume that in each step, a fraction of the particles moves randomly to the right or to the left. If \( f_k(t) \) is the energy of particles in cell \( k \) at time \( t \), then the energy of particles at time \( t + 1 \) is proportional to \( f_k(t) \) plus \( f_{k-1}(t) - f_k(t) \) to the right and \( f_k(t) - f_{k+1}(t) \) to the left. This is a discrete version of the second derivative because \( dx^2 f_{xx}(x, t) \approx (f(x+dx, t) - 2f(x, t) + f(x-dx, t)) \).

TO THE DERIVATION OF THE WAVE EQUATION. A wave can be modeled by \( n \) particles linked by springs. Assume that the water particles move up and down only. If \( f_i(t) \) is the height of the particles, then the right particle pulls with a force \( f_{i+1} - f_i \) and the left particle with a force \( f_{i-1} - f_i \). Again, \( f_{i-1}(t) - 2f_i(t) + f_{i+1}(t) \) is a discrete version of the second derivative \( f_{xx} \). By Newtons law, the acceleration \( f_{tt}(t, x) \) at position \( x \) is proportional to \( f_{xx} \).

TO THE DERIVATION OF BURGERS EQUATION. Assume that \( \mu = 0 \) for a moment. If the wave \( f \) has height close to \( c \), we see that \( f(x, t) + cf_t(x, t) = 0 \) which has the solution \( f(x, t) = f(x - ct, 0) \). The waves travel forward with a speed which depends on the height of the wave. Higher waves travel faster. The additional term \( \mu f_{xx} \) plays the same role as in the heat equation: the potential energy, which is proportional to the height of the wave, dissipates into the neighborhood.

VISUALIZATION. We can plot the wave height \( f(x, t) \) as a function of \( x \) for different but fixed times \( t \).

\[ f(x, 0) \quad f(x, 1) \quad f(x, 2) \quad f(x, 3) \quad f(x, 4) \]