• Start by printing your name in the above box and check your section in the box to the left.

• Do not detach pages from this exam packet or unstaple the packet.

• Please write neatly. Answers which are illegible for the grader can not be given credit.

• No notes, books, calculators, computers, or other electronic aids can be allowed.

• You have 90 minutes time to complete your work.

• The hourly exam itself will have space for work on each page. This space is excluded here in order to save printing resources.
1) T F The length of the sum of two vectors is always the sum of the length of the vectors.

Solution:
There is a triangle inequality in general. But equality only holds for parallel vectors pointing in the same direction.

2) T F For any three vectors, $\vec{v} \times (\vec{w} + \vec{u}) = \vec{w} \times \vec{v} + \vec{u} \times \vec{v}$.

Solution:
The cross product is distributive but not commutative.

3) T F The set of points which satisfy $x^2 + 2x + y^2 - z^2 = 0$ is a cone.

Solution:
$x^2 + y^2 - z^2 = 0$ is a cone. Completion of the square adds another constant and the surface is a one-sheeted hyperboloid.

4) T F The functions $\sqrt{x+y-1}$ and $\log(x+y-1)$ have the same domain of definition.

Solution:
The square root is defined for 0, but the logarithm is not defined at 0.

5) T F If $P, Q, R$ are 3 different points in space that don’t lie in a line, then $\vec{PQ} \times \vec{RQ}$ is a vector orthogonal to the plane containing $P, Q, R$.

Solution:
The vectors $\vec{PQ}$ and $\vec{RQ}$ are both in the plane. The cross product is perpendicular to the plane.

6) T F The line $\vec{r}(t) = \langle 1 + 2t, 1 + 3t, 1 + 4t \rangle$ hits the plane $2x + 3y + 4z = 9$ at a right angle.
Solution:
The vector $\langle 2, 3, 4 \rangle$ is in the line and perpendicular to the plane.

7) T   F

The graph of $f(x, y) = \cos(xy)$ is a level surface of a function $g(x, y, z)$.

Solution:
Yes, it is the surface $g(x, y, z) = c$ for the function $g(x, y, z) = z - \cos(xy)$ and the constant $c = 0$.

8) T   F

For any two vectors, $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$.

Solution:
The cross product is anti commutative.

9) T   F

If $|\vec{v} \times \vec{w}| = 0$ for all vectors $\vec{w}$, then $\vec{v} = \vec{0}$.

Solution:
Assume $\vec{v}$ is not $\vec{0}$, then take $\vec{w}$ as a vector which is perpendicular to $\vec{v}$.

10) T   F

If $\vec{u}$ and $\vec{v}$ are orthogonal vectors, then $(\vec{u} \times \vec{v}) \times \vec{u}$ is parallel to $\vec{v}$.

Solution:
The vector in question is perpendicular to $\vec{u}$ and perpendicular to $\vec{u} \times \vec{v}$. Also $\vec{v}$ is perpendicular to $\vec{u}$ and $\vec{v}$.

11) T   F

Every vector contained in the line $\vec{r}(t) = \langle 1 + 2t, 1 + 3t, 1 + 4t \rangle$ is parallel to the vector $\langle 1, 1, 1 \rangle$.

Solution:
The line contains the point $(1, 1, 1)$ and a vector $\langle 2, 3, 4 \rangle$. 
The curvature of the curve $2\vec{r}(4t)$ at $t = 0$ is twice the curvature of the curve $\vec{r}(t)$ at $t = 0$.

**Solution:**
The curvature of the first curve is 1/2 of the curvature of the second curve.

The set of points which satisfy $x^2 - 2y^2 - 3z^2 = 0$ form an ellipsoid.

**Solution:**
The surface is an elliptical cone.

If $\vec{v} \times \vec{w} = (0, 0, 0)$, then $\vec{v} = \vec{w}$.

**Solution:**
The two vectors can be parallel and nonzero.

Every vector contained in the line $\vec{r}(t) = (1 + 2t, 1 + 3t, 1 + 4t)$ is parallel to the vector $(1, 1, 1)$.

**Solution:**
It is parallel to $(2, 3, 4)$

Two nonzero vectors are parallel if and only if their cross product is $\vec{0}$.

**Solution:**
You can use the formula $|\vec{v} \times \vec{w}| = |\vec{v}| |\vec{w}| \sin(\alpha)$. If this is zero, then either one of the vectors is the zero vector or $\sin(\alpha) = 0$. In all cases, this can be considered parallel.

The function $u(x, t) = x^2/2 + t$ satisfies the heat equation $u_t = u_{xx}$.

**Solution:**
Just differentiate.
18) **T** F  Any function of three variables \( f(x, y, z) \) satisfies the partial differential equation \( f_{xyz} + f_{yxz} = 2f_{zxy} \).

**Solution:**
By Clairot’s theorem

19) **T** F  If \( f_x(x, y) = f_y(x, y) \) for all \( x, y \), then \( f(x, y) \) is a constant.

**Solution:**
\( f_x = f_y \) is an example of a PDE called a transport equation. It has solutions like for example \( f(x, y) = x + y \). Any function which stays invariant by replacing \( x \) with \( y \) is a solution: like \( f(x, y) = \sin(xy) + x^5y^5 \).

20) **T** F  The value of the function \( f(x, y) = \sin(-x + 2y) \) at \((0.001, -0.002)\) can by linear approximation be estimated as \(-0.003\).

**Solution:**
The correct approximation would be \( f(0,0) + 0.001(-1) - 0.002(2) = -0.005 \).
Problem 2a) (5 points)

Match the contour maps with the corresponding functions $f(x, y)$ of two variables. No justifications are needed.

<table>
<thead>
<tr>
<th>Enter I,II,III,IV,V or VI here</th>
<th>Function $f(x, y)$</th>
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<tbody>
<tr>
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Problem 2b) (5 points)

Match the parametric surfaces with their parameterization. No justification is needed.
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<tr>
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<tr>
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<tr>
<td></td>
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Solution:

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Surface I is a graph.
Surface II is a surface of revolution.
Surface III is algebraic. One of the traces is \((u^3, u^2)\), an other trace is the parabola \((v^2, v)\).
Surface IV is a plane.

Problem 3) (10 points)

Use the technique of linear approximation to estimate \(f(\log(2) + 0.001, 0.006)\) for \(f(x, y) = e^{2x-y}\). (Here, log means the natural logarithm).

Solution:

\[
L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)
\]
\[
f(x_0, y_0) = e^{2 \log 2} = 4
\]
\[
f_x(x_0, y_0) = 8
\]
\[
f_y(x_0, y_0) = -4
\]
\[
L(x, y) = 4 + 0.001 \cdot 8 - 4 \cdot 0.006 = 3.984.
\]

Problem 4) (10 points)
Consider the equation
\[ f(x, y) = 2y^3 + x^2y^2 - 4xy + x^4 = 0 \]
It defines a curve, which you can see in the picture. Near the point \( x = 1, y = 1 \), the function can be written as a graph \( y = y(x) \). Find the slope of that graph at the point \((1, 1)\).

Solution:
Use the formula for implicit differentiation which is derived from the chain rule \( f_x(x, y(x)) \cdot 1 + f_y(x, y(x)) \cdot y'(x) = 0 \). The slope is \( y'(x) = -f_x(x, y)/f_y(x, y)(x,y) = (1, 1) = -1/2 \).
An other possibility to solve this problem is to find the equation of the tangent line which is \( f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) = 0 \) and find the slope \( m \) by writing this equation as \( y = mx + b \). It gives of course the same result.

Problem 5) (10 points)
a) (6 points) Find a parameterization of the line of intersection of the planes \( 3x - 2y + z = 7 \) and \( x + 2y + 3z = -3 \).

b) (4 points) Find the symmetric equations
\[
\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}
\]
representing that line.

Solution:
a) The line of intersection has the direction \( (3, -2, 1) \times (1, 2, 3) = 8(-1, -1, 1) \). The parameterization is \( \mathbf{r}(t) = (1, -2, 0) + t(-1, -1, 1) \).

b) If a line contains the point \((x_0, y_0, z_0)\) and a vector \(\langle a, b, c \rangle\), then the symmetric equation is
\[
\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.
\]
In our case, where \((x_0, y_0, z_0) = (1, -2, 0)\) and \(\langle a, b, c \rangle = (-1, -1, 1)\), the symmetric equations are \( x - 1 = y + 2 = -z \).

Problem 6) (10 points)
a) (4 points) Find the area of the parallelogram with vertices \( P = (1, 0, 0), Q = (0, 2, 0), R = (0, 0, 3) \) and \( S = (-1, 2, 3) \).

b) (3 points) Verify that the triple scalar product has the property \([\vec{u}+\vec{v}, \vec{v}+\vec{w}, \vec{w}+\vec{u}] = 2[\vec{u}, \vec{v}, \vec{w}]\).

c) (3 points) Verify that the triple scalar product \([\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})\) has the property
\[
|[\vec{u}, \vec{v}, \vec{w}]| \leq ||\vec{u}|| \cdot ||\vec{v}|| \cdot ||\vec{w}||
\]

**Solution:**
a) One has to realize which vectors form the sides of the parallelogram. The solution is \( |\vec{PQ} \times \vec{PR}| = 7 \).

b) \([u + v, v + w, w + u] = [u, v, w] + [u, v, u] + [u, w, u] + [v, v, u] + [v, w, u] + [v, w, w] + [v, v, u] \). Any term, where two parallel vectors appear is zero. So, only \( 2[u, v, w] \) remains on the right hand side.

c) Build the parallelepiped spanned by \( u, v, w \) and note that one can shear it in such a way that it is contained in the box of size \( ||\vec{u}|| \) and \( ||\vec{v}|| \) and \( ||\vec{w}|| \). You can also see the identity by using angle formulas for the dot product \( \vec{v} \cdot \vec{w} = ||\vec{v}|| ||\vec{w}|| \cos(\alpha) \) and the length of the cross product \( ||\vec{v} \times \vec{w}|| = ||\vec{v}|| ||\vec{w}|| \sin(\beta) \)
\[
|[\vec{u}, \vec{v}, \vec{w}]| \leq ||\vec{u}|| ||\vec{v}|| ||\vec{w}|| \cos(\alpha) ||\vec{v}|| \sin(\beta)\]
where \( \beta \) is the angle between \( \vec{v} \) and \( \vec{w} \) and where \( \alpha \) is the angle \( \vec{v} \times \vec{w} \) and \( \vec{u} \).

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**Problem 7) (10 points)**

Find the distance between the two lines
\[
\vec{r}_1(t) = \langle t, 2t, -t \rangle
\]
and
\[
\vec{r}_2(t) = \langle 1 + t, t, t \rangle .
\]

**Solution:**
The point \( P = (0, 0, 0) \) is on the first line. The point \( Q = (1, 0, 0) \) on the second line. The vector \( \vec{v} = \langle 1, 2, -1 \rangle \) in the first line and \( \vec{w} = \langle 1, 1, 1 \rangle \) in the second line. We have \( \vec{n} = \langle 3, -2, -1 \rangle \). Now, the distance is \( 3/\sqrt{14} \). \( (Q - P) \cdot \vec{n}/||\vec{n}|| = \langle 1, 0, 0 \rangle \cdot \langle 3, -2, -1 \rangle / ||\vec{n}|| = 3/\sqrt{14} \).
Find an equation for the plane that passes through the origin and whose normal vector is parallel to the line of intersection of the planes $2x + y + z = 4$ and $x + 3y + z = 2$.

**Solution:**
The line of intersection is parallel to the crossed product of $\vec{v} = \langle 2, 1, 1 \rangle$ and $\vec{w} = \langle 1, 3, 1 \rangle$ which is $\langle -2, -1, 5 \rangle$. This vector is perpendicular to the plane we are looking for. The equation of the plane is $-2x - y + 5z = 0$.

**Problem 9** (10 points)

The intersection of the two surfaces $x^2 + \frac{y^2}{2} = 1$ and $z^2 + \frac{y^2}{2} = 1$ consists of two curves.

a) (4 points) Parameterize each curve in the form $\vec{r}(t) = (x(t), y(t), z(t))$.
b) (3 points) Set up the integral for the arc length of one of the curves.
c) (3 points) What is the arc length of this curve?

**Solution:**
a) Fix first $x(t), y(t)$ to satisfy the first equation then get $z(t) = \cos(t)$ by solving the second equation for $z$. $\vec{r}(t) = (\cos(t), \sqrt{2} \sin(t), \pm \cos(t))$.
b) We find the velocity $\vec{r}'(t) = (-\sin(t), \sqrt{2} \cos(t), -\sin(t))$ and then the speed $|\vec{r}'(t)| = \sqrt{\sin^2(t) + 2 \cos^2(t) + \sin^2(t)} = \sqrt{2}$. The length is $\int_0^{2\pi} |\vec{r}'(t)| \, dt = \int_0^{2\pi} \sqrt{2} \, dt$. Also an expression like $\int_0^{2\pi} \sqrt{\sin^2(t) + 2 \cos^2(t) + \sin^2(t)} \, dt$ is here correct at this stage.
c) Evaluate the integral $2\sqrt{2}\pi$.

**Problem 10** (10 points)

a) (6 points) Find the curvature $\kappa(t)$ of the space curve $\vec{r}(t) = (-\cos(t), \sin(t), -2t)$ at the point $\vec{r}(0)$.

b) (4 points) Find the curvature $\kappa(t)$ of the space curve $\vec{r}(t) = (-\cos(5t), \sin(5t), -10t)$ at the point $\vec{r}(0)$. 
**Hint.** Use one of the two formulas for the curvature

\[ \kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}, \]

where \( \vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \). The curvatures in b) can be derived from the curvature in a). There is no need to redo the calculation, but we need a justification.

**Solution:**

a) We use the second formula for the curvature: \( \vec{r}'(t) = \langle \sin(t), \cos(t), -2 \rangle \). \( \vec{r}''(t) = \langle \cos(t), -\sin(t), 0 \rangle \). The speed of the curve satisfies \( |\vec{r}'(t)| = \sqrt{5} \). The vector \( \vec{r}'(t) \times \vec{r}''(t) \) is \( (-2\sin(t), -2\cos(t), -1) \) which has length \( \sqrt{5} \). Therefore, the curvature is constant \( \kappa(t) = 1/5 \).

b) Because the curvature is independent of the parametrization, the curvature is again \( 1/5 \).