5/23/2006, FORTH PRACTICE FINAL **Math 21a, Spring 2006**

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- Please mark the box to the left which lists your section.
- Do not detach pages from this exam packet or unstaple the packet.
- Show your work. Answers without reasoning can not be given credit except for the True/False and multiple choice problems.
- Please write neatly.
- Do not use notes, books, calculators, computers, or other electronic aids.
- Unspecified functions are assumed to be smooth and defined everywhere unless stated otherwise.
- You have 180 minutes time to complete your work.
- Biochem sections can ignore problems with vector fields and line integrals this semester.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10

11A		10
12A		10
13A		10

11B		10
12B		10
13B		10

Total:		140

Problem 1) True/False questions (20 points)

1) T F

The projection vector $\text{proj}_{\vec{v}}(\vec{w})$ is parallel to $\vec{w}$.

Solution:
It is parallel to $\vec{v}$.

2) T F

Any parametrized surface S is a graph of a function $f(x,y)$.

Solution:
A counter example is the sphere.

3) T F

If the directional derivatives $D_{\vec{v}}(f)(1,1)$ and $D_{\vec{w}}(f)(1,1)$ are both 0 for $\vec{v} = \langle 1,1\rangle/\sqrt{2}$ and $\vec{w} = \langle 1,-1\rangle/\sqrt{2}$, then $(1,1)$ is a critical point.

Solution:
Indeed $\nabla f(1,1)$ must be perpendicular to $\vec{v}$ and $\vec{w}$ and so be the zero vector.

4) T F

The linearization $L(x,y)$ of $f(x,y) = x + y + 4$ at $(0,0)$ satisfies $L(x,y) = f(x,y)$.

Solution:
The linearization of any linear function at $(0,0)$ is the function itself.

5) T F

For any function $f(x,y)$ of two variables, the line integral of the vector field $\vec{F} = \nabla f$ on a level curve $\{f = c\}$ is always zero.

Solution:
The gradient is perpendicular to the velocity vector.

6) T F

If $\vec{F}$ is a vector field of unit vectors defined in $1/2 \leq x^2 + y^2 \leq 2$ and $\vec{F}$ is tangent to the unit circle C, then $\int_C \vec{F} \cdot d\vec{r}$ is either equal to 2π or -2π.

Solution:
$\vec{r}'$ is parallel to $\vec{F}$ so that $\vec{F} \cdot \vec{r}'$ is equal to 1 or -1.

7) | T | F |

If a curve C intersects a surface S at a right angle, then at the point of intersection, the tangent vector to the curve is parallel to the normal vector of the surface.

Solution:
This is clear once you know what the question means.

8) | T | F |

The curvature of the curve $\vec{r}(t) = \langle \cos(3t), \sin(6t) \rangle$ at the point $\vec{r}(0)$ is smaller than the curvature of the curve $\vec{r}(t) = \langle \cos(30t), \sin(60t) \rangle$ at the point $\vec{r}(0)$.

Solution:
The curvature is independent of the parametrization of the curve.

9) | T | F |

At every point (x, y, z) on the hyperboloid $x^2 + y^2 - z^2 = 1$, the vector $\langle x, y, -z \rangle$ is tangent to the hyperboloid.

Solution:
It is normal to the hyperboloid.

10) | T | F |

The set $\{\phi = \pi/2, \theta = \pi\}$ in spherical coordinates is the negative x axis.

Solution:
$\phi = \pi/2$ forces us to be on the xy-plane. $\theta = \pi$ is the negative x axis

11) | T | F |

The integral $\int_0^1 \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin^2(\phi) \; d\phi \, d\theta \, d\rho$ is equal to the volume of the unit ball.

Solution:
If $\sin^2(\phi)$ would be $\sin(\phi)$, then it would be the integral in spherical coordinates.

12) T F

Four points A, B, C, D are located in a single common plane if $(B - A) \cdot ((C - A) \times (D - A)) = 0$.

Solution:
This is the volume of the parallelepiped with corners A, B, C, D. If the volume is zero, then the parallelepiped is flat and the points in a plane.

13) T F

If a function $f(x, y)$ has a local maximum at $(0, 0)$, then the discriminant D is negative.

Solution:
False, we also can have $D = 0$ like for $f(x, y) = 1 - x^4 - y^4$.

14) T F

The integral $\int_0^x \int_y^1 f(x, y)\, dx\, dy$ represents a double integral over a bounded region in the plane.

Solution:
The integral is not properly defined. There can be no variable in the most outer integral.

15) T F

The following identity is true: $\int_0^3 \int_0^{2x} x^2\, dydx = \int_0^6 \int_{y/2}^3 x^2\, dx\, dy$

Solution:
Make a picture and draw the triangle.

TF problems 16-20 are for regular and physics sections only:

16) T F

The integral $\iint_S \mathrm{curl}(\vec{\mathrm{F}}) \cdot d\vec{\mathrm{S}}$ over the surface S of a cube is zero for all vector fields $\vec{F}$.

Solution:
By the divergence theorem, this flux integral is equal to the triple integral of $\mathrm{div}(\vec{F})$ over the cube. But since $\mathrm{div}(\mathrm{curl}(\vec{F})) = 0$, this integral is zero.

17) T F A vector field $\vec{F}$ defined on three space which is incompressible ($\mathrm{div}(\vec{F}) = 0$) and irrotational ($\mathrm{curl}(\vec{F}) = 0$) can be written as $\vec{F} = \nabla f$ with $\Delta f = \nabla^2 f = 0$.

Solution:
Every gradient field $F = \nabla f$, for which $\Delta(f) = 0$, is also incompressible.

18) T F If a vector field $\vec{F}$ is defined at all points of three-space except the origin, and $\mathrm{curl}(\vec{F}) = \vec{0}$ everywhere, then the line integral of $\vec{F}$ around the circle $x^2 + y^2 = 1$ in the xy-plane is equal to zero.

Solution:
The circle is the boundary of a hemisphere which is contained in the region, where $\vec{F}$ is defined.

19) T F The identity $\mathrm{curl}(\mathrm{grad}(\mathrm{div}(\vec{F}))) = \vec{0}$ is true for all vector fields $\vec{F}(x, y, z)$.

Solution:
Already $\mathrm{curl}(\mathrm{grad}(f)) = \vec{0}$.

20) T F If $\vec{F} = \mathrm{curl}(\vec{G})$, where $\vec{G} = \langle e^{e^x}, 5^x z^5, \sin y \rangle$, then $\mathrm{div}(\vec{F}(x, y, z)) > 0$ for all (x, y, z).

Solution:
$\mathrm{div}(\mathrm{curl}(\vec{F})) = 0$.

TF problems 21-25 are for biochem sections only:

21) T F

The expected value of the sum of two random variables is the sum of their expected values.

Solution:
This is the linearity of summation or integration.

22) T F

Let X be a random variable. Suppose that we know both the expectation $E(X)$ and and variance $\mathrm{D}(X)$. Does this information determine $\mathrm{E}(X^2 - 5X + 4)$?

Solution:
Yes, by the linearity of the expectation, we can write $\mathrm{E}[X^2 - 5X + 4] = \mathrm{E}[X^2] - 5\mathrm{E}[X] + 4$. Because $\mathrm{D}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$.

23) T F

If A and B are two events and B has positive probability, then $P(A|B)$ is always less than or equal to $P(A)$.

Solution:
$P(A|B) = P(A \cap B)/P(B)$. If $A = B$, then $P(A|B) = 1$. This is possible with $P(A) < 1$.

24) T F

The function $\Phi_\xi = \frac{1}{1+x^2}$ is the distribution function of some random variable ξ.

Solution:
It is not a distribution function. It is not even a density function.

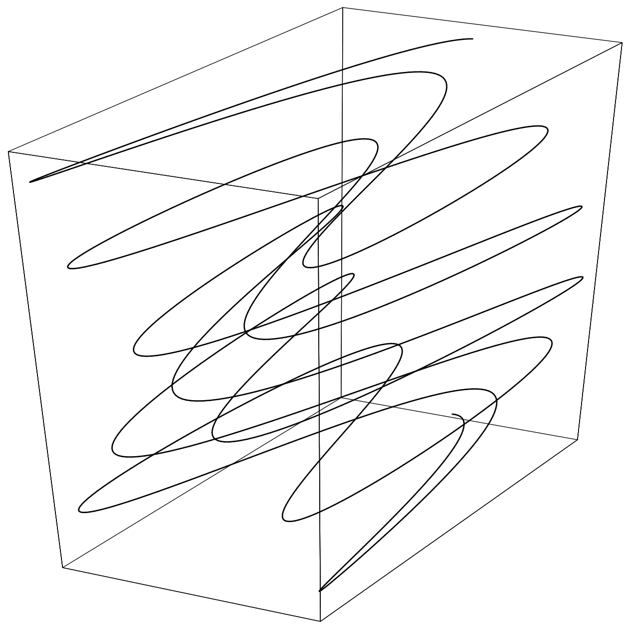
25) T F

Suppose you throw two fair dice. The probability that the sum of their upturned faces is 11 is 2/36.

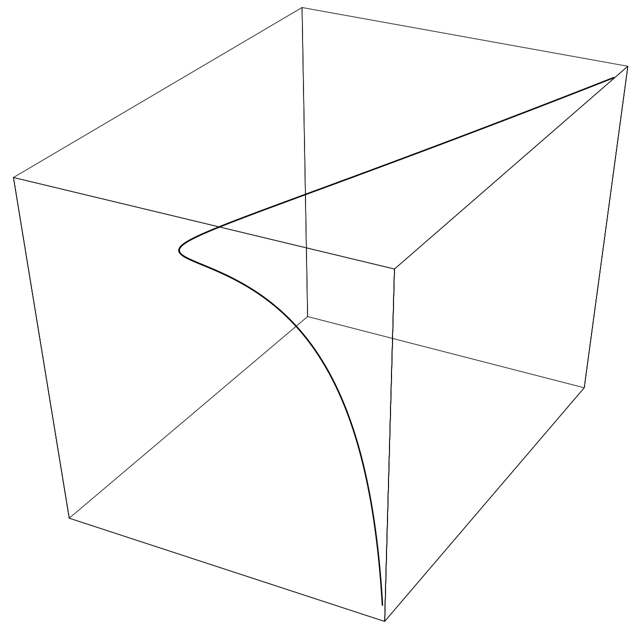
Solution:
The two events are independent. There are 36 elements in the probability space. There are two possibilities to show 11. Therefore, the probability is 2/36.

Problem 2) (10 points)

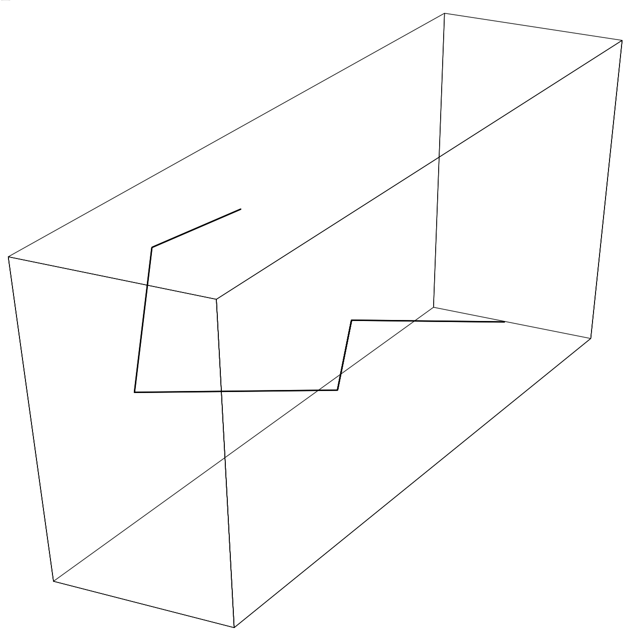
Match the equations with the space curves. No justifications are needed.



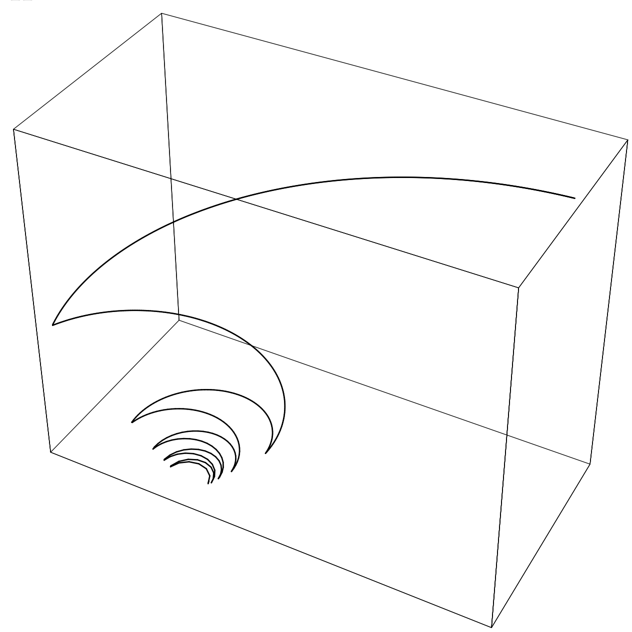
I



II



III



IV

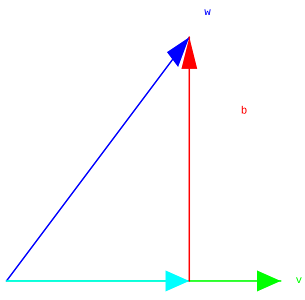
Enter I,II,III,IV here	Equation
	$\vec{r}(t) = \langle t^2, t^3 - t, t \rangle$
	$\vec{r}(t) = \langle \|1 - \|t\|\|, \|t - \|t-1\|\|, t \rangle$
	$\vec{r}(t) = \langle 2\sin(5t), \cos(11t), t \rangle$
	$\vec{r}(t) = \langle t\sin(1/t), t\|\cos(1/t)\|, t \rangle$

Solution:

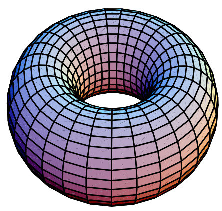
Enter I,II,III,IV here	Equation
II	$\vec{r}(t) = \langle t^2, t^3 - t, t \rangle$
III	$\vec{r}(t) = \langle \lvert 1 - \lvert t \rvert \rvert, \lvert t - \lvert t-1 \rvert \rvert, t \rangle$
I	$\vec{r}(t) = \langle 2\sin(5t), \cos(11t), t \rangle$
IV	$\vec{r}(t) = \langle t\sin(1/t), t\lvert\cos(1/t)\rvert, t \rangle$

Problem 3) (10 points)

Match the equations with the objects. No justifications are needed.



I



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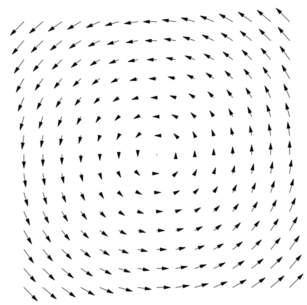

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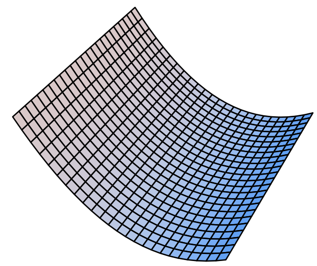

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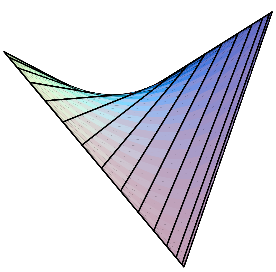

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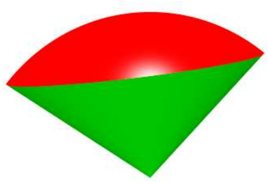

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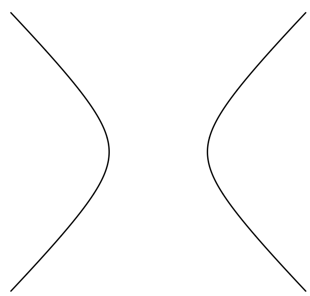
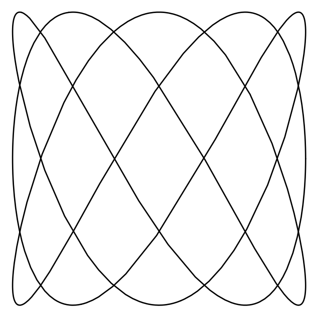


VII



VIII

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
	$\vec{r}(s,t) = \langle (2+\cos(s))\cos(t), (2+\cos(s))\sin(t), \sin(s)\rangle$
	$\vec{r}(t) = \langle \cos(3t), \sin(5t)\rangle$
	$x^2 + y^2 - z^2 = 1$
	$\vec{F}(x,y,z) = \langle -y, x, 1\rangle$
	$x^2 + y^2 + z^2 \leq 1, x^2 + y^2 \leq z^2, z \geq 0$
	$z = f(x,y) = x^2 - y$
	$g(x,y) = x^2 - y^2 = 1$
	$\vec{F}(x,y) = \langle -y, x\rangle$

Solution:

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
II	$\vec{r}(s,t) = \langle (2+\cos(s))\cos(t), (2+\cos(s))\sin(t), \sin(s)\rangle$
VIII	$\vec{r}(t) = \langle \cos(3t), \sin(5t)\rangle$
V	$x^2 + y^2 - z^2 = 1$
I	$\vec{F}(x,y,z) = \langle -y, x, 1\rangle$
VI	$x^2 + y^2 + z^2 \leq 1, x^2 + y^2 \leq z^2, z \geq 0$
IV	$z = f(x,y) = x^2 - y$
VII	$g(x,y) = x^2 - y^2 = 1$
III	$\vec{F}(x,y) = (-y, x)$

Problem 4) (10 points)

a) Find an equation for the plane Σ passing through the points $\vec{r}(0), \vec{r}(1), \vec{r}(2)$, where $\vec{r}(t) = \langle t^2, t^4, t\rangle$.

b) Find the distance between the point $\vec{r}(-1)$ and the plane Σ found in a).

Solution:
a) $r(0) = (0,0,0), r(1) = (1,1,1), r(2) = (4,16,2)$. The normal vector is $n = (a,b,c) = (1,1,1) \times (4,16,2) = (-14,-2,12)$. The plane has the equation $ax + by + cz = d$, where d is obtained from plugging in one of the points. The answer is $\boxed{-14x - 2y - 12z = 0.}$.
b) $Q = (0,0,0)$ is a point on the plane. The point $P = \vec{r}(-1) = (1,1,-1)$ has distance $|\vec{PQ} \cdot \vec{n}|/|\vec{n}| = (1,1,-1) \cdot (-14,-2,12)/\sqrt{196+4+144} = \boxed{28/\sqrt{344}}$ from the plane.

Problem 5) (10 points)

A vector field $\vec{F}(x,y)$ in the plane is given by $\vec{F}(x,y) = \langle x^2+5, y^2-1 \rangle$. Find all the critical points of $|\vec{F}(x,y)|$ and classify them. At which point or points is $|\vec{F}(x,y)|$ minimal?

Solution:
Extremize $f(x,y) = x^4 + 10x^2 + 25 + y^4 - 2y^2 + 1$. The gradient is $\nabla f(x,y) = \langle 4x^3 + 20x, 4y^3 - 4y \rangle = \langle 4x(x^2+20), 4y(y^2-1) \rangle$. We have $\nabla f(x,y) = \langle 0,0 \rangle$ for the critical points $(0,-1)$, $(0,0)$, and $(0,1)$. We have $f_{xx} = 20$ at all three points and $D(0,0) = -80, D(0,\pm 1) = 160$. The point $(0,0)$ is a saddle point and the two critical points $(0,1), (0,-1)$ are the global minima.

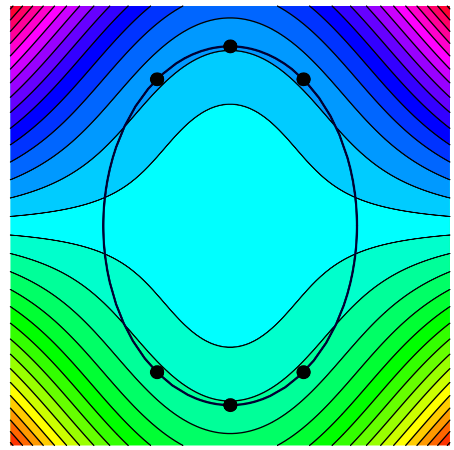
Problem 6) (10 points)

A house is situated at the point $(0,0)$ in the middle of a mountainous region. The altitude at each point (x,y) is given by the equation $f(x,y) = 4x^2y + y^3$. There is a pathway in the shape of an ellipse around the house, on which the (x,y) coordinates satisfy $2x^2 + y^2 = 6$. Find the highest and lowest points in the closed region bounded by the path.

Solution:
The gradient of f is $\nabla f = \langle 8xy, 4x^2 + 3y^2 \rangle$. From the second equation, the only critical point for the function without constraints is $x = y = 0$. At this critical point, the function f has value 0. If $g(x,y) = 2x^2 + y^2$, then $\nabla g = \langle 4x, 2y \rangle$. $\boxed{(0,0) \text{ is a critical point with } f(0,0) = 0.}$
For local maxima and minima on the ellipse, the equation $\nabla f = \lambda \nabla g$ must hold, so

$$\langle 8xy, 4x^2 + 3y^2 \rangle = \langle 4x\lambda, 2y\lambda \rangle.$$

Equating the first coordinates gives $8xy = 4x\lambda$, so $x = 0$ or $\lambda = 2y$. In the first case, $y^2 = 6$, so $y = \pm\sqrt{6}$. Therefore the function f has value $\pm 6\sqrt{6}$ in this case. Otherwise, $\lambda = 2y$, and so equating the second coordinates, $2y\lambda = 2y \cdot 2y = 4y^2 = 4x^2 + 3y^2$. Hence $y^2 = 4x^2$. At this point, one can work out $y = \pm 2x$. In any event, $2x^2 + y^2 = 6x^2$, so $6x^2 = 6$. Therefore $x = \pm 1$ and $y = \pm 2$.
In conclusion, there are 6 constraint extrema on the boundary $(0, 6\sqrt{6}), (0, -\sqrt{6}), (1, 2), (1, -2), (-1, 2), (-1, -2)$
The maximal value of f is $+16$ and obtained at $\boxed{\text{the points } (1,2) \text{ and } (-1,2)}$.



Problem 7) (10 points)

a) (4 points) Where does the tangent plane at $(1,1,1)$ to the surface $z = e^{x-y}$ intersect the z axis?

b) (4 points) Estimate $f(x,y,z) = 1 + \log(1 + x + 2y + z) + 2\sqrt{1+z}$ at the point $(0.02, -0.001, 0.01)$.

c) (2 points) $f(x,y,z) = 0$ defines z as a function $g(x,y)$ of x and y. Find the partial derivative $g_x(x,y)$ at the point $(x,y) = (0,0)$.

Solution:
a) The tangent plane has the equation $ax + by + cz = d$, where $\langle a, b, c\rangle = \nabla g(1, 1, 1)$, where $g(x, y, z) = z - e^{x-y}$. Because $\nabla g(x, y, z) = \langle -e^{x-y}, e^{x-y}, 1\rangle$, we have $\nabla g(1, 1, 1) = \langle -1, 1, 1\rangle$. The plane has the equation $-x + y + z = d$. The number d can be obtained by plugging in the point $(1, 1, 1)$. Therefore the plane is $-x + y + z = 1$. This plane intersects the z axes at $\boxed{z = 1}$.
b) The linearization of $1 + \log(1 + x + 2y + z) + 2\sqrt{1 + z}$ at $(0, 0, 0)$ is $L(x, y, z) = 3 + x + 2y + 2z$. Now $L(0.02, -0.001, 0.01) = 3 + 0.02 - 2 \cdot 0.001 + 2 * 0.1 \boxed{= 3.038}$.
c) The implicit computation formula is $g_x(x, y) = -f_x(x, y, z)/f_z(x, y, z)$. We have $f_z = (1/(1+x+2y+z)+1/\sqrt{1 + z})$ and $f_x = 1/(1+x+2y+z)$ so that $g_x(0, 0) = -1/(\sqrt{1 + z}+1)$. The value of z would have to be computed numerically.

Problem 8) (10 points)

For each of the following quantities, set up a double or triple integral using any coordinate system you like. You do not have to evaluate the integrals, but the bounds of each single integral must be specified explicitly.

1. (3 points) The volume of the tetrahedron with vertices $(0, 0, 0)$, $(3, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 3)$.
2. (4 points) The surface area of the piece of the paraboloid $z = x^2 + y^2$ lying in the region $z = x^2 + y^2$, where $0 \leq z \leq 1$.
3. (3 points) The volume of the solid bounded by the planes $z = -1$, $z = 1$ and the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$.

Solution:
(1) The tetrahedron is bounded by the xy, yz, zx-planes and the plane $z = 3 - x - y$. The triple integral would be: $\int_0^3 \int_0^{3-x} \int_0^{3-x-y} \; dz \; dy \; dx$. Evaluating the inner integral $\int_0^3 \int_0^{3-x} (3 - x - y) \, dydx$.
(2) The parameterization $\vec{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), r^2 \rangle$ gives

$$\begin{aligned} |\vec{r}_r \times \vec{r}_\theta| &= |\langle \cos(\theta), \sin(\theta), 2r \rangle \times \langle -r\sin(\theta), r\cos(\theta), 0 \rangle| \\ &= |\langle \; -2r^2\cos(\theta), -2r^2\sin(\theta), r \rangle| = \sqrt{4r^4 + r^2} \; . \end{aligned}$$

The surface area integral is $\int_0^{2\pi} \int_0^1 \sqrt{4r^4 + r^2} \; dr \; d\theta$ or $\int_0^1 \int_0^{2\pi} \sqrt{4r^4 + r^2} \; d\theta \; dr$.
Parameterization $\vec{r}(x, y) = \langle \; x, \; y, \; x^2 + y^2 \; \rangle$ gives

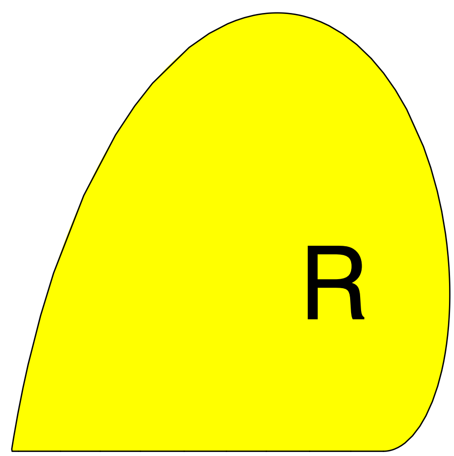
$$|\vec{r}_x \times \vec{r}_y| = |\langle 1, 0, 2x \; \rangle \times \langle 0, 1, 2y \; \rangle| = |\langle -2x, -2y, 1 \rangle| = \sqrt{4x^2 + 4y^2 + 1} \; ,$$

and the inttegral $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{4x^2 + 4y^2 + 1} \; dxdy$ or $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{4x^2 + 4y^2 + 1} \; dydx$.
(3) Using cylindrical coordinates, the integral is $\int_{-1}^1 \int_0^{\sqrt{1+z^2}} \int_0^{2\pi} r \quad d\theta dr \quad dz$ or $2\pi \cdot \int_{-1}^1 \int_0^{\sqrt{1+z^2}} r dr \; dz$ or $2\pi \Big(\int_0^1 \int_{-1}^1 dz \; dr \; + 2 \int_1^{\sqrt{2}} \int_{\sqrt{r^2-1}}^1 \; r dz \; dr \Big)$. Using spherical coordinates, one would have to split up the integral into two parts.

Problem 9) (10 points)

A region R in the xy-plane is given in polar coordinates by $r(\theta) \leq \theta$ for $\theta \in [0, \pi]$. You see the region in the picture to the right. Evaluate the double integral

$$\iint_R \frac{\cos(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}(\pi - \sqrt{x^2 + y^2})} \; dx \; dy \; .$$



Solution:
The region becomes a triangle in polar coordinates. Setting up the integral with $dA = dr d\theta$ does not work. The integral $\int_0^\pi \int_0^\theta \frac{\cos(r)}{r(\pi - r)} r dr d\theta$ can not be solved. We have to change the order of integration:

$$\int_0^\pi \int_r^\pi \frac{\cos(r)}{r(\pi - r)} r \; d\theta \; dr$$

Evaluating the inner integral gives $\int_0^\pi \cos(r) \; dr = \boxed{0}$.

Problem 10) (10 points)

A car drives up a freeway ramp which is parametrized by

$$\vec{r}(t) = \langle \cos(t), 2\sin(t), t \rangle, \qquad 0 \le t \le 3\pi \ .$$

a) (3 points) Set up an integral which gives the length of the ramp. You do need not need to evaluate it.

b) (3 points) Find the unit tangent vector $\vec{T}$ to the curve at the point where $t = 0$.

c) (4 points) Suppose the wind pattern in the area is such that the wind exerts a force $\vec{F} = \langle 4x^2, y, 0 \rangle$ on the car at the position (x, y, z). What is the total work done by the car against the wind as it drives up the ramp?

Solution:
a) $\vec{r}\,'(t) = \langle \sin(t), 2\cos(t), 1 \rangle$ and $|\vec{r}\,'| = \sqrt{\sin^2(t) + 4\cos^2(t) + 1}$. The integral is

$$\int_0^{3\pi} \sqrt{\sin^2(t) + 4\cos^2(t) + 1} \; dt \ .$$

=

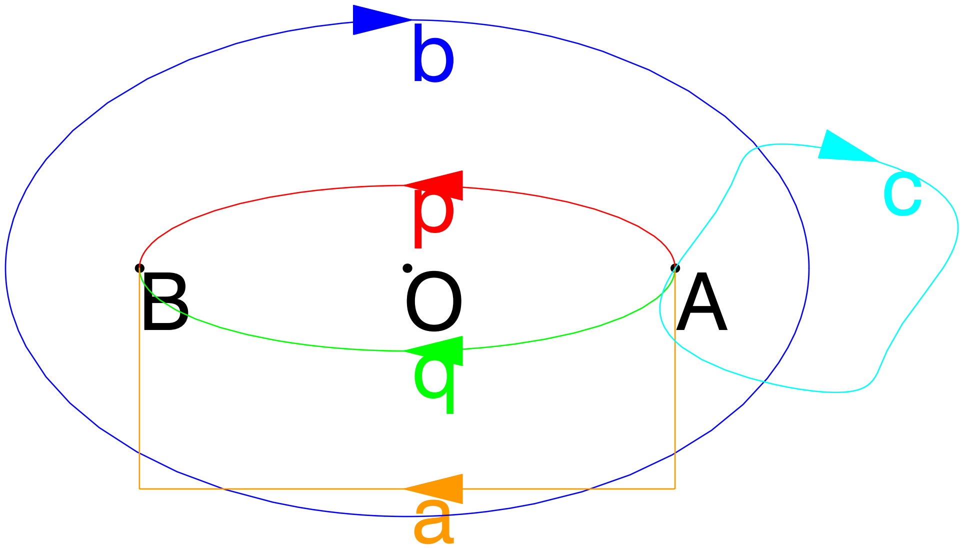
$$\int_0^{3\pi} \sqrt{2 + 3\cos^2(t) + 1} \; dt \ .$$

b) $\vec{T}(t) = \vec{r}(t)/|\vec{r}(t)| = \langle \sin(t), 2\cos(t), 1 \rangle / \sqrt{\sin^2(t) + 4\cos^2(t) + 1}$. At $t = 0$, we have $\vec{T}(t) = \langle 0, 2, 1 \rangle / \sqrt{5}$.
c) The line integral is $\int_0^{3\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \; dt$. Note that $\vec{F}$ is a gradient field with potential function $f(x, y, z) = 4x^3/3 + y^2/2$ so that the line integral is $f(\vec{r}(3\pi)) - f(\vec{r}(0)$ by the fundamental theorem of line integrals. Now $f(\vec{r}(3\pi)) = f(-1, 0, 3\pi) = -4/3$ and $f(\vec{r}(0)) = f(1, 0, 0) = 4/3$, so that the result is $\boxed{-8/3}$.

The following problem 11A is for regular and physics sections only:

Problem 11 A) (10 points)

Suppose $\vec{F}$ is an irrotational vector field in the plane (that is, its curl is everywhere zero) that is not defined at the origin $O = (0, 0)$. Suppose the line integral of $\vec{F}$ along the path p from A to B is 5 and the line integral of $\vec{F}$ along the path q from A to B is -4. Find the line integral of $\vec{F}$ along the following three paths:



a) (3 points) The path a from A to B going clockwise below the origin.

b) (4 points) The closed path b encircling the origin in a clockwise direction.

c) (3 points) The closed path c starting at A and ending in A without encircling the origin.

Solution:
a) The result is the same for the path a and the path q. The vector field is conservative in the lower half plane. The result is $\boxed{-4}$. b) The line integral is the same as the difference of the line integral along q and the line integral along p which is $-4-5 = \boxed{-9}$. The path $q-p$ encircles the origin in the same direction than the path b. Because the curl is 0 in the region enclosed by these two curves, Greens theorem assures that the line integrals are the same.
c) The vector field F is conservative in the righth half plane. By the fundamental theorem of line integrals or using the closed loop property, the result is $\boxed{0}$.

The following problem 12A is for regular and physics sections only:

Problem 12 A) (10 points)

Let S be the surface which bounds the region enclosed by the paraboloid $z = x^2 + y^2 - 1$ and the xy plane. Let $\vec{F}$ be the vector field $\vec{F}(x, y, z) = \langle x + e^{\sin(z)}, z, -y \rangle$.

a) (5 points) Find the flux of $\vec{F}$ through the surface S.

b) (5 points) Find the flux of $\vec{F}$ through the part of the surface S that belongs to the paraboloid, oriented so that the normal vector points downwards.

Solution:
a) The flux out of the whole surface can be determined by the Divergence Theorem. The flux over the whole surface is the integral of the divergence over the solid region it encloses. The divergence of the vector field is 1. So the flux is the volume of the solid region, namely the integral $\int_D 1 - x^2 - y^2 \; dydx$ over the unit disc D. Converting to polar coordinates, this is $\int_0^{2\pi} \int_0^1 (1-r^2) r \; dr \; d\theta = \int_0^{2\pi} \int_0^1 r - r^3 \; dr \; d\theta = \int_0^{2\pi} \frac{1}{2} - \frac{1}{4} \; d\theta = \int_0^{2\pi} \frac{1}{4} \; d\theta = \frac{\pi}{2}$. So the flux out of the whole surface is $\boxed{\pi/2}$.
b) There the outward unit normal vector to the bottom surface is $\vec{k}$, so the flux is

$$\int_D -y \; dA = \int_0^{2\pi} \int_0^1 -r \sin\theta \; r \; dr \; d\theta = \int_0^{2\pi} -\frac{1}{3} \sin\theta \; d\theta = 0 \; .$$

Therefore the flux out of the "roof" D is $\boxed{0}$. The flux through the paraboloid part is $\pi/2 - 0 = \boxed{\pi/2}$. We use a) and the Divergence theorem.

The following problem 13A is regular and physics sections only:

Problem 13 A) (10 points)

Let $\vec{F}$ be the vector field $\vec{F}(x, y, z) = \langle 4z + \cos(\cos x), y^2, x + 2y \rangle$.

a) (4 points) Let C be the curve given by the parameterization $\vec{r}(t) = \langle \cos t, 0, \sin t \rangle$, for $0 \leq t \leq 2\pi$. Find the line integral of $\vec{F}$ along C.

b) (6 points) Let S be the hemisphere of the unit sphere defined by $y \leq 0$. Find the flux of the curl of $\vec{F}$ out of S. In other words, find

$$\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} \; .$$

For part b), the surface S is oriented so that the normal vector has a positive y-component.

Solution:
a) The curl of the vector field is $\langle 2,3,0\rangle$. The parameterization describes the circle $x^2+z^2=1$, where $y=0$. The curve starts at $(1,0,0)$ and rotates towards back towards $(0,0,1)$. By Stokes theorem, the line integral can be computed as the flux of curl(F) through the unit disk D in the xz plane which has the normal vector $r_u \times r_v = -\vec{j}$ and curl$(\vec{F})(x,y,z)\cdot(r_u \times r_v) = -3$. The flux is

$$\int\int_D -3 \; dx \; dz = -3\pi \; .$$

b) The boundary of S is the curve C, oriented as in part a). So the answer is also -3π, again by $\boxed{\textbf{Stokes theorem}}$. (It was also possible to invoke the **divergence theorem** to conclude that the flux through D and the flux through S are the same, because the difference between the two fluxes is the triple integral of div$(\vec{F})$ over that solid, which is 0.)

The following problem 11 B is for biochem sections only:

Problem 11 B) (10 points)

A laboratory uses a certain test for HIV. If a patient is infected, the test will give a positive result with probability 95% and a negative result with probability 5%. If the patient is not infected, the test gives a negative result with probability 99%.

a) (3 points) A person who is not infected is tested ten times. What is the probability that at least two of the tests will be positive?

b) (4 points) The Centers for Disease Control estimate that 0.4 % of Americans are HIV-positive. An American is chosen randomly and tested for HIV. The test gives a positive result. What is the probability that this person does not have HIV?

c) (3 points) The person is tested again, and again the test gives a positive result. What is the probability that the person does not have HIV?

Solution:
a) The probability for a noninfected person to have a positive result is $p = 0.01$. The probability to have 2 positive results in 10 tests is

$$\binom{10}{2} p^2(1-p)^8 = 45 * 0.01^2 * 0.99^8 \ .$$

b) Let A be the event of having HIV and let B be the event to be tested positively. We have to compute $P(A|B)$. We know $P(A) = 0.004$ and $P(B|A) = 0.95$ and $P(B|A^c) = 0.01$. From

$$0.95 = P(B|A) = P(B \cap A)/P(A) = P(B \cap A)/0.004$$

and

$$0.01 = P(B|A^c) = P(B \cap A^c)/P(A^c) = P(B \cap A^c)/0.996$$

we get $P(B) = P(B \cap A) + P(B \cap A^c) = P(B|A)P(A) + P(B|A^c)P(A^c) == 0.004 * 0.95 + 0.01 * 0.996$. This is the probabity that the test gives a positive result. Now we can compute $P(A^c|B) = P(A^c \cap B)/P(B) = 0.01 * 0.996/(0.004 * 0.95 + 0.01 * 0.996)$.
c) Let C be the event that the test gives twice a positive result. We do not know the probabilty of C directly but we can compute it using conditional probabilities: we know $P(C|A) = P(B|A)^2 = 0.95^2$ and $P(C|A^c) = P(B|A^c)^2 = (0.01)^2$. Now, $P(C) = P(C \cap A) + P(C \cap A^c) = P(C|A)P(A) + P(C|A^c)P(A^c) = \boxed{0.95^2 \cdot 0.004 + (0.01)^2 \cdot 0.996} = 0.0037096$.

The following problem 12 B is for biochem sections only:

Problem 12 B) (10 points)

Adam has to wait for his sister Sally and mother Mary to be ready before he can head to the football game. Let S be the random variable equal to the waiting time for Sally, which has density function $p_S(x) = \begin{cases} e^{-x} & if\ x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$
Let M be the random variable equal to the waiting time for Mary, which has density function $p_M(x) = \begin{cases} 2e^{-2x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$

a) (3 points) What is the expectation and variance of of S?

b) (3 points) Calculate the distribution functions $\Phi_S(x)$ and $\Phi_M(x)$.

c) (4 points) Let X be the random variable which is equal to the time when Adam can leave, that is, until both Sally and Mary are ready. Assume that S and M are independent random variables. What is the probability density function $p_X(x)$ of X?

Solution:
a) The expectation is $E(S) = \int_0^\infty xe^{-x}\ dx = \boxed{1}$. The variance is $E(S^2) - E(S)^2 = \int_0^\infty x^2e^{-x}\ dx = 2 - 1 = \boxed{1}$.
b) $\Phi_S(x) = \int_0^x e^{-t}\ dt = 1 - e^{-x}$ and $\int_0^x 2e^{-2t}\ dt = \boxed{1 - e^{-2x}}$.
c) $\Phi_X(x)$ the probability that Adam can leave before time x is equal to the product of the probabilities that Sally is ready and the probability that Mary is ready. So $\Phi_X(x) = \Phi_S(x)\Phi_M(x) = (1 - e^{-x})(1 - e^{-2x}) \boxed{= 1 - e^{-x} - e^{-2x} + e^{-3x}, x \geq 0}$.
Therefore, $p_X(x) = \frac{d}{dx}\Phi_X(x) = (-3 + 2 * e^x + e^{2x})/e^{3x} = \boxed{e^{-x} + 2e^{-2x} - 3e^{-3x},\ x \geq 0}$.

The following problem 13 B is for biochem sections only:

Problem 13 B) (10 points)

There are 2 white balls and 3 blue balls in a bag. We select two balls from the bag, without replacing the first ball after selecting it. Let A be the event that the first selected ball is white, and let B be the event that exactly one of the selected balls is blue.
a) (4 points) Compute $P(A)$, $P(B)$ and $P(A|B)$.

b) (3 points) Are A and B independent events?

c) (3 points) What is the expectation of the random variable X giving the total number of blue balls selected?

Solution:
$\Omega = \{ww, wb, bb, bw\}$ with probabilities $p(ww) = (2/5)(1/4) = 1/10, p(wb) = (2/5)(3/4) = 3/10, p(bb) = (3/5)(2/4) = 3/10, p(bw) = (3/5)(2/4) = 3/10$.
a) We have $A = \{ww, wb\}$ and $P(A) = 1/10 + 3/10\ \boxed{2/5}$.
We have $B = \{wb, bw\}$ and $P(B) = 3/10 + 3/10 = 3/5$. We have $A \cap B = \{wb\}$ and $P(A \cap B) = 3/10$. Now $P(A|B) = P(A \cap B)/P(B) = (3/10)/(3/5) = \boxed{1/2}$.
b) In order to have independent events, we would need $3/10 = P(A \cap B) = P(A) \cdot P(B) = (2/5)(3/5) = 6/25$. No, the events are not independent.
c) $X(ww) = 2, X(wb) = X(bw) = 1, X(bb) = 0$. The expectation is $E(X) = 2P(bb) + 1P(bw) + 1P(wb) + 0P(bb) = 2 \cdot (3/5)(2/4) + 1 \cdot (3/10 + 3/10) + 0 \cdot (2/5)(1/4) = \boxed{6/5}$.