coordinates and vectors in the plane and in space
\[ v = (v_1, v_2, v_3), w = (w_1, w_2, w_3), v + w = (v_1 + w_1, v_2 + w_2, v_3 + w_3) \]
dot product \( v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3 = |v||w| \cos(\alpha) \)
cross product, \( v \times w = 0, w \times v = 0, |v \times w| = |v||w| \sin(\alpha) \)
triple cross product \( u \cdot (v \times w) \) volume of parallelepiped
parallel vectors \( v \times w = 0 \), orthogonal vectors \( v \cdot w = 0 \)
scalar projection \( \text{comp}_w(v) = v \cdot w/|w| \), vector projection \( \text{proj}_w(v) = (v \cdot w)w/|w|^2 \)
completion of square technique: example \( x^2 - 4x + y^2 = 1 \) is equivalent to \( (x - 2)^2 + y^2 = -3 \)
distance \( d(P, Q) = |PQ| = \sqrt{(P_1 - Q_1)^2 + (P_2 - Q_2)^2 + (P_3 - Q_3)^2} \)

1. Geometry of Space

symmetric equation of line \( \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \)
plane \( ax + by + cz = d \)
parametric equation for line \( \vec{x} = \vec{x}_0 + t\vec{v} \)
parametric equation for plane \( \vec{x} = \vec{x}_0 + t\vec{v} + s\vec{w} \)
switch from parametric to implicit descriptions for lines and planes
domain and range of functions \( f(x, y) \)
graph \( G = \{(x, y, f(x, y))\} \)
intercepts: intersections of \( G \) with coordinate axes
traces: intersections with coordinate planes
generalized traces: intersections with \( \{x = c\}, \{y = c\} \) or \( \{z = c\} \)
quadrics: ellipsoid, paraboloid, hyperboloids, cylinder, cone, parabolic hyperboloid
plane \( ax + by + cz = d \) has normal \( \vec{n} = (a, b, c) \)
line \( \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \) contains \( \vec{v} = (a, b, c) \)
sets \( g(x, y, z) = c \) describe surfaces, example graphs \( g(x, y, z) = z - f(x, y) \)
linear equation \( 2x + 3y + 5z = 7 \) defines plane
quadratic equation \( x^2 - 2y^2 + 3z^2 = 4 \) defines quadric surface
distance point-plane: \( d(P, \Sigma) = |(\vec{PQ}) \cdot \vec{n}|/|\vec{n}| \)
distance point-line: \( d(P, L) = |(\vec{PQ}) \times \vec{u}|/|\vec{u}| \)
distance line-line: \( d(L, M) = |(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|/|\vec{u} \times \vec{v}| \)
finding plane through three points \( P, Q, R \) find first normal vector
3. Curves

plane and space curves $\mathbf{r}(t)$
velocity $\mathbf{r}'(t)$, Acceleration $\mathbf{r}''(t)$
unit tangent vector $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$
unit normal vector $\mathbf{N}(t) = \mathbf{T}'(t)/|\mathbf{T}'(t)|$
bivector normal vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$
curvature $\kappa(t) = |\mathbf{T}'(t)|/|\mathbf{r}'(t)|$
arc length $\int_a^b |\mathbf{r}'(t)| \, dt$
$\mathbf{r}'(t)$ is tangent to the curve
$\tilde{v} = \mathbf{r}'$ then $\mathbf{r} = \int_0^t \tilde{v} \, dt + \mathbf{c}$
$\kappa(t) = (\mathbf{r}'(t) \times \mathbf{r}''(t))/|\mathbf{r}'(t)|^3$,
\[ \frac{d}{dt}(\mathbf{v}(t) \cdot \mathbf{\tilde{w}}(t)) = \mathbf{\tilde{v}}(t) \cdot \mathbf{\tilde{w}}(t) + \mathbf{v}(t) \cdot \mathbf{\tilde{w}}''(t) \]
$\mathbf{T}, \mathbf{N}, \mathbf{B}$ are unit vectors which are perpendicular to each other
find parameterizations of basic curves (i.e. intersections of surfaces)

4. Surfaces

polar coordinates $(x, y) = (r \cos(\theta), r \sin(\theta))$
cylindrical coordinates $(x, y, z) = (r \cos(\theta), r \sin(\theta), z)$
spherical coordinates $(x, y, z) = (\rho \cos(\phi) \sin(\theta), \rho \sin(\theta) \sin(\phi), \rho \cos(\phi))$
g$(r, \theta) = 0$ polar curve, especially $r = f(\theta)$, polar graphs
g$(r, \theta, z) = 0$ cylindrical surface, especially $r = f(z, \theta)$ or $r = f(z)$ surface of revolution
g$(\rho, \theta, \phi) = 0$ spherical surface especially $\rho = f(\theta, \phi)$
f$(x, y) = c$ level curves of $f(x, y)$
g$(x, y, z) = c$ level surfaces of $g(x, y, z)$
circle: $x^2 + y^2 = r^2$, $\mathbf{r}(t) = (r \cos(t), r \sin(t))$.
ellipse: $x^2/a^2 + y^2/b^2 = 1$, $\mathbf{r}(t) = (a \cos(t), b \sin(t))$
sphere: $x^2 + y^2 + z^2 = r^2$, $\mathbf{r}(u, v) = (r \cos(u) \sin(v), r \sin(u) \sin(v), r \cos(v))$
ellipsoid: $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, $\mathbf{r}(u, v) = (a \cos(u) \sin(v), b \sin(u) \sin(v), c \cos(v))$
line: $ax + by = d$, $\mathbf{r}(t) = (t, d/b - ta/b)$
plane: $ax + by + cz = d$, $\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{v} + v\mathbf{u}$, $(a, b, c) = \mathbf{v} \times \mathbf{u}$
surface of revolution: $r(\theta, z) = f(z)$, $\mathbf{r}(u, v) = (f(v) \cos(u), f(v) \sin(u), v)$
graph: $g(x, y, z) = z - f(x, y) = 0$, $\mathbf{r}(u, v) = (u, v, f(u, v))$

5. Partial Derivatives

$f_x(x, y) = \frac{\partial}{\partial x} f(x, y)$ partial derivative
partial differential equation PDE: $F(f, f_x, f_t, f_{xx}, f_{tt}) = 0$
f$_t = f_{xx}$ heat equation
$f_{tt} - f_{xx} = 0$ 1D wave equation
$f_x - f_t = 0$ transport equation
$f_x f_t - f_{xt} = 0$ Burger equation
$f_{xx} + f_{yy} = 0$ Laplace equation
$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0)$ linear approximation
tangent line: $L(x, y) = L(x_0, y_0)$, $ax + by = d$ with $a = f_x(x_0, y_0), b = f_y(x_0, y_0), d = ax_0 + by_0$
tangent plane: $L(x, y, z) = L(x_0, y_0, z_0)$
estimate $f(x, y, z)$ by $L(x, y, z)$ near $(x_0, y_0, z_0)$
f$(x, y)$ differentiable if $f_x, f_y$ are continuous
$f_{xy} = f_{yx}$ Clairot’s theorem
$\mathbf{r}'_u(u, v), \mathbf{r}'_v$ tangent to surface $\mathbf{r}(u, v)$
6. Chain Rule

\[ \nabla f(x, y) = (f_x, f_y), \quad \nabla f(x, y, z) = (f_x, f_y, f_z), \quad \text{gradient} \]

\[ D_v f = \nabla f \cdot v \quad \text{directional derivative} \]

\[ \frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \quad \text{chain rule} \]

\[ \nabla f(x_0, y_0, z_0) \text{ is orthogonal to the level surface } f(x, y, z) = c \text{ which contains} \]

\[ (x_0, y_0, z_0). \]

\[ \frac{d}{dt} f(\vec{x} + t\vec{v}) = D_v f \text{ by chain rule} \]

\[ \frac{\partial}{\partial x} f(x_0, y_0, z_0) = \frac{y-y_0}{z-z_0} = \frac{z-z_0}{y-y_0} \quad \text{normal line to surface } f(x, y, z) = c \text{ at} \]

\[ (x_0, y_0, z_0) \]

\[ (x - x_0)f_x(x_0, y_0, z_0) + (y - y_0)f_y(x_0, y_0, z_0) + (z - z_0)f_z(x_0, y_0, z_0) = 0 \quad \text{tangent plane at} \]

\[ (x_0, y_0, z_0) \]

directional derivative is maximal in the \( \vec{v} = \nabla f \) direction

\[ f(x, y) \text{ increases, if we walk on the } xy\text{-plane in the } \nabla f \text{ direction} \]

partial derivatives are special directional derivatives

if \( D_v f(\vec{x}) = 0 \) for all \( \vec{v} \), then \( \nabla f(\vec{x}) = \vec{0} \)

implicit differentiation: \( f(x, y(x)) = 0, f_x 1 + f_y y'(x) = 0 \) gives \( y'(x) = -f_x/f_y \)

7. Extrema

\[ \nabla f(x, y) = (0, 0), \quad \text{critical point or stationary point} \]

\[ D = f_{xx} f_{yy} - f_{xy}^2, \quad \text{discriminant or Hessian determinant} \]

\[ f(x_0, y_0) \geq f(x, y) \text{ in a neighborhood of } (x_0, y_0) \text{ local maximum} \]

\[ f(x_0, y_0) \leq f(x, y) \text{ in a neighborhood of } (x_0, y_0) \text{ local minimum} \]

\[ \nabla f(x, y) = \lambda \nabla g(x, y), g(x, y) = c, \lambda \text{ Lagrange multiplier} \]

two constraints: \( \nabla f = \lambda \nabla g + \mu \nabla h, g = c, h = d \)

Second derivative test: \( \nabla f = (0, 0), D > 0, f_{xx} < 0 \text{ local max}, \nabla f = (0, 0), D > 0, f_{xx} > 0 \text{ local min} \)

\[ \nabla f = (0, 0), D < 0 \text{ saddle} \]

8. Double Integrals

\[ \int \int_R f(x, y) \, dA \quad \text{double integral} \]

\[ \int_a^b \int_c^d \, f(x, y) \, dy \, dx \quad \text{integral over rectangle} \]

\[ \int_a^b \int_{g_1(x)}^{g_2(x)} \, f(x, y) \, dy \, dx \quad \text{type I region} \]

\[ \int_c^d \int_{h_1(y)}^{h_2(y)} \, f(x, y) \, dx \, dy \quad \text{type II region} \]

\[ \int \int_R f(r, \theta) r \, dr \, d\theta \quad \text{polar coordinates} \]

\[ \int \int_R \nabla f \times \nabla \phi \, dudv \quad \text{surface area} \]

\[ \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy \quad \text{Fubini} \]

\[ \int \int_R 1 \, dx \, dy \quad \text{area of region } R \]

\[ \int \int_R f(x, y) \, dx \, dy \quad \text{volume of solid bounded by } \text{graph(f) xy-plane} \]

9. Triple Integrals

\[ \int \int \int_R f(x, y, z) \, dV \quad \text{triple integral} \]

\[ \int_a^b \int_c^d \int_{g_1(x, y)}^{g_2(x, y)} \, f(x, y, z) \, dz \, dy \, dx \quad \text{integral over rectangular box} \]

\[ \int_a^b \int_{h_1(x, y)}^{h_2(x, y)} \, f(x, y) \, dz \, dy \, dx \quad \text{type I region} \]

\[ f(r, \theta, z) = |r| \, dr \, d\theta \, dz \quad \text{cylindrical coordinates} \]

\[ \int \int \int_R f(r, \theta, z) \, r^2 \sin(\phi) \, dr \, d\theta \, d\phi \quad \text{spherical coordinates} \]

\[ \frac{\partial}{\partial x} \] for polar and cylindrical \( (x, y, z) = (r \cos \theta, r \sin \theta, z) \) \( \text{Jacobian of } T(u, v) = (x(u, v), y(u, v)) \)

\[ \frac{\partial}{\partial x} \] for spherical \( (x, y, z) = (r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi) \) \( \text{Jacobian of } T(u, v, w) = (x, y, z) \)

\[ \int_b^a \int_c^d \int_v^u \, f(x, y, z) \, dz \, dy \, dx \quad \text{Fubini} \]

\[ V = \int \int \int_R 1 \, dV \quad \text{volume of solid } R \]

\[ M = \int \int \int_R \rho(x, y, z) \, dV \quad \text{mass of solid } R \text{ with density } \rho \]

\[ (\int \int \int_R x \, dV, \int \int \int_R y \, dV, \int \int \int_R z \, dV) \quad \text{center of mass} \]


10. Line Integrals

\[ F(x, y) = (P(x, y), Q(x, y)) \text{ vector field in the plane} \]
\[ F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z)) \text{ vector field in space} \]
\[ \int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) \, dt \text{ line integral} \]
\[ F(x, y) = \nabla f(x, y) \text{ gradient field} = \text{ potential field = conservative} \]
\[ f \text{ is the potential of } F \]
\[ \text{Fundamental thm of line int: } \int_C F \cdot dr = \int_{r(b)}^{r(a)} f \]
\[ \text{For smooth gradient fields in simply connected region } R \int_C F \, dr = 0, \text{ for all closed curves } C \]

11. Green’s Theorem

\[ F(x, y) = (P, Q), \text{ curl}(F) = Q_x - P_y = \nabla \times F, \text{ div}(F) = P_x + Q_y = \nabla \cdot F \]
\[ F(x, y, z) = (P, Q, R), \text{ curl}(P, Q, R) = (R_y - Q_z, P_z - R_x, Q_x - P_y) = \nabla \times F, \text{ div}(P, Q, R) = P_x + Q_y + R_z = \nabla \cdot F \]
\[ \Delta f = \text{divgrad}(f) = f_{xx} + f_{yy} + f_{zz} \text{ Laplacian for functions} \]
\[ \Delta F = \Delta(P, Q, R) = (\Delta P, \Delta Q, \Delta R) \text{ Laplacian for vector fields} \]
\[ \nabla = (\partial_x, \partial_y, \partial_z), \text{ grad}(F) = \nabla f, \text{curl}(F) = \nabla \times F, \text{ div}(F) = \nabla \cdot F \]
\[ \Delta f = \nabla \cdot \nabla f \]
\[ \text{Green’s theorem: } C \text{ boundary of } R, \text{ then } \int_C F \cdot dr = \int_R \text{curl}(F) \, dxdy \]
\[ \text{div(curl}(F)) = 0 \]
\[ \text{curl(grad}(F)) = \hat{0} \]
\[ \text{curl(curl}(F)) = \text{grad(div}(F)) - \Delta(F) \]

12. Stokes and Divergence Theorem

\[ F(x, y, z) \text{ vector field, } S = r(R) \text{ parametrized surface} \]
\[ r_u \times r_v \text{ normal vector, } \vec{n} = \frac{r_u \times r_v}{|r_u \times r_v|} \text{ unit normal vector} \]
\[ |r_u \times r_v| \, du dv = dS \text{ surface element} \]
\[ r_u \times r_v \, du dv = d\vec{S} = \vec{n}dS \text{ normal surface element} \]
\[ \int_S f \, dS = \int_S f(r(u, v)) |r_u \times r_v| \, du dv \text{ surface integral} \]
\[ \int_S F \cdot d\vec{S} = \int_S F(r(u, v)) \cdot (r_u \times r_v) \, du dv \text{ flux integral} \]
\[ \text{Stokes’s theorem: } C \text{ boundary of surface } S, \text{ then } \int_C F \cdot dr = \int_S \text{curl}(F) \cdot dS \]
\[ \text{Divergence theorem: } E \text{ bounded by } S \text{ then } \int_E F \cdot dV = \int_S \text{div}(F) \, dS \]