SURFACE AREA

\[
\int \int_R |\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \, dudv
\]

is the area of the surface.

INTEGRAL OF A SCALAR FUNCTION ON A SURFACE. If \( S \) is a surface, then \( \int \int_S f(x,y) \, dS \) should be an average of \( f \) on the surface. If \( f(x,y) = 1 \), then \( \int \int_S dS \) should be the area of the surface. If \( S \) is the image of \( \vec{r} \) under the map \( (u,v) \mapsto \vec{r}(u,v) \), then \( dS = |\vec{r}_u \times \vec{r}_v| \, dudv \).

DEFINITION. Given a surface \( S = \vec{r}(R) \), where \( R \) is a domain in the plane and where \( \vec{r}(u,v) = (x(u,v), y(u,v), z(u,v)) \). The surface integral of \( f(u,v) \) on \( S \) is defined as

\[
\int \int_S f(u,v) \, dS = \int \int_R f(u,v)|\vec{r}_u \times \vec{r}_v| \, dudv.
\]

INTERPRETATION. If \( f(x,y) \) measures a quantity then \( \int \int_S f \, dS \) is the average of the function \( f \) on \( S \).

EXPLANATION OF \( |\vec{r}_u \times \vec{r}_v| \). The vector \( \vec{r}_u \) is a tangent vector to the curve \( u \mapsto \vec{r}(u,v) \), when \( v \) is fixed and the vector \( \vec{r}_v \) is a tangent vector to the curve \( v \mapsto \vec{r}(u,v) \), when \( u \) is fixed. The two vectors span a parallelogram with area \( |\vec{r}_u \times \vec{r}_v| \). A little rectangle spanned by \([u,u+du]\) and \([v,v+dv]\) is mapped by \( \vec{r} \) to a parallelogram spanned by \([\vec{r}', \vec{r} + \vec{r}_u]\) and \([\vec{r}', \vec{r} + \vec{r}_v]\).

A simple case: consider \( \vec{r}(u,v) = (2u,3v,0) \). This surface is part of the xy-plane. The parameter region \( R \) just gets stretched by a factor 2 in the \( x \) coordinate and by a factor 3 in the \( y \) coordinate. \( \vec{r}_u \times \vec{r}_v = (0,0,6) \) and we see for example that the area of \( \vec{r}(R) \) is 6 times the area of \( R \).

POLAR COORDINATES. If we take \( \vec{r}(u,v) = (u \cos(v), u \sin(v), 0) \), then the rectangle \([0, R] \times [0, 2\pi]\) is mapped into a flat surface which is a disk in the \( xy \)-plane. In this case \( \vec{r}_u \times \vec{r}_v = (\cos(v), \sin(v), 0) \times (-u \sin(v), u \cos(v), 0) = (0,0,u) \) and \(|\vec{r}_u \times \vec{r}_v| = u = r \). We can explain the integration factor \( r \) in polar coordinates as a special case of a surface integral.

THE AREA OF THE SPHERE.

The map \( r : (u,v) \mapsto (L \cos(u) \sin(v), L \sin(u) \sin(v), L \cos(v)) \) maps the rectangle \([0, 2\pi] \times [0, \pi]\) onto the sphere of radius \( L \). We compute \( \vec{r}_u \times \vec{r}_v = L \sin(v) \vec{r}(u,v) \). So, \(|\vec{r}_u \times \vec{r}_v| = L^2 |\sin(v)|\) and \( \int \int_R 1 \, dS = \int_0^{2\pi} \int_0^\pi L^2 \sin(v) \, dvdu = 4\pi L^2 \).

SURFACE AREA OF GRAPHS. For surfaces \((u,v) \mapsto (u,v,f(u,v)),\) we have \( \vec{r}_u = (1,0,f_u(u,v)) \) and \( \vec{r}_v = (0,1,f_v(u,v)) \). The cross product \( \vec{r}_u \times \vec{r}_v = (-f_u,-f_v,1) \) has the length \( \sqrt{1+f_u^2+f_v^2} \). The area of the surface above a region \( R \) is \( \int \int_R \sqrt{1+f_u^2+f_v^2} \, dA \).

EXAMPLE. The surface area of the paraboloid \( z = f(x,y) = x^2 + y^2 \) is (use polar coordinates) \( \int_0^{2\pi} \int_0^r \sqrt{1+4r^2} \, r \, dr \, d\theta = 2\pi(2/3)(1+4r^2)^{3/2}/8 \big|_0^1 = \pi(5^{3/2} - 1)/6 \).
AREA OF SURFACES OF REVOLUTION. If we rotate the graph of a function \( f(x) \) on an interval \([a,b]\) around the \( x \)-axes, we get a surface parameterized by \((u,v) \mapsto \tilde{r}(u,v) = (v, f(v) \cos(u), f(v) \sin(u))\) on \( R = [0, \pi] \times [a,b] \) and is called a surface of revolution. We have \( \tilde{r}_u = (0, -f(v) \sin(u), f(v) \cos(u)), \tilde{r}_v = (1, f'(v) \cos(u), f'(v) \sin(u)) \) and \( \tilde{r}_u \times \tilde{r}_v = (-f(v)f'(v), f(v) \cos(u), f(v) \sin(u)) = f(v)(1 - f'(v) \cos(u), \sin(u)) \) which has the length \( |\tilde{r}_u \times \tilde{r}_v| = |f(v)| \sqrt{1 + f'(v)^2} \).

EXAMPLE. If \( f(x) = x \) on \([0,1]\), we get the surface area of a cone: \( \int_0^{2\pi} \int_0^1 x \sqrt{1 + 1/x^4} \, dv \, du = 2\pi \sqrt{2}/2 = \pi \sqrt{2} \).

P.S. In computer graphics, surfaces of revolutions are constructed from a few prescribed points \((x_i, f(x_i))\). The machine constructs a function \((\text{spline})\) and rotates

GABRIEL’S TRUMPET. Take \( f(x) = 1/x \) on the interval \([1, \infty)\).

**Volume:** The volume is (use cylindrical coordinates in the \( x \)-direction): \( \int_1^\infty \pi f(x)^2 \, dx = \pi \int_1^\infty 1/x^2 \, dx = \pi \).

**Area:** The area is \( \int_0^{2\pi} \int_0^\infty 1/x \sqrt{1 + 1/x^4} \, dx \geq 2\pi \int_1^\infty 1/x \, dx = 2\pi \log(x)|_1^\infty = \infty \).

The Gabriel trumpet is a surface of finite volume but with infinite surface area! You can fill the trumpet with a finite amount of paint, but this paint does not suffice to cover the surface of the trumpet!

**Question.** How long does a Gabriel trumpet have to be so that its surface is 500 cm\(^2\) (area of sheet of paper)? Because \( 1 \leq \sqrt{1 + 1/x^4} \leq \sqrt{2} \), the area for a trumpet of length \( L \) is between \( 2\pi \int_1^L 1/x \, dx = 2\pi \log(L) \) and \( \sqrt{22}\pi \log(L) \). In our case, \( L \) is between \( e^{500/(\sqrt{22}\pi)} \sim 2 \times 10^{24} \text{cm} \) and \( e^{500/(2\pi)} \sim 4 \times 10^{34} \text{cm} \). Note that the universe is about 10\(^{26}\) cm long (assuming that the universe expanded with speed of light since 15 Billion year). It could not accommodate a Gabriel trumpet with the surface area of a sheet of paper.

MÖBIUS STRIP. The surface \( \tilde{r}(u,v) = (2+v \cos(u/2) \cos(u), (2+v \cos(u/2)) \sin(u), v \sin(u/2)) \) parametrized by \( R = [0, 2\pi] \times [-1,1] \) is called a Möbius strip.

The calculation of \( |\tilde{r}_u \times \tilde{r}_v| = 4 + 3v^2/4 + 4v \cos(u/2) + v^2 \cos(u)/2 \) is straightforward but a bit tedious. The integral over \([0,2\pi] \times [-1,1]\) is \( 17\pi \).

**Question.** If we build the Moebius strip from paper. What is the relation between the area of the surface and the weight of the surface?

**Remarks.**
1) An OpenGL implementation of an Escher theme can be admired with "xlock -inwindow -mode moebius" on an X-terminal. 2) A patent was once assigned to the idea to use a Moebius strip as a conveyor belt. It would last twice as long as an ordinary one.