ICE: NASH’S PROBLEM

In this ICE, you solve Nash’s problem, he gave to a multivariable calculus class. Remember Nash saying in the movie "A beautiful mind":

"It might take some of you a few months to solve it, for most of you however it might take a life time".

NASH’s PROBLEM. Find a subset $X$ of $\mathbb{R}^3$ with the property that if $V$ is the set of vector fields $F$ on $\mathbb{R}^3 \setminus X$ which satisfy $\text{curl}(F) = 0$ and $W$ is the set of vector fields $F$ which are conservative: $F = \nabla f$. Then, the space $V/W$ should be 8 dimensional.

Remark. The meaning of the last sentence means that there should be 8 vector fields $F_i$ which are not gradient fields and which have vanishing curl outside $X$. (You might learn more about dimensions in Math21b, linear algebra.) Furthermore, you should not be able to write any of the 8 vector fields as a sum of multiples of the other 7 vector fields.

You actually saw a two dimensional version of the problem in class:

2D VERSION OF NASH’s PROBLEM.

If $X = \{0\}$ and $V$ is the set of vector fields $F$ on $\mathbb{R}^2 \setminus X$ which satisfy $\text{curl}(F) = 0$ and $W$ is the set of vector fields $F$ which are gradient fields, then $\dim(V/W) = 1$.

The vector field $F$ is $F(x,y) = (-y/(x^2 + y^2), x/(x^2 + y^2))$ is a gradient field in $\mathbb{R}^2 \setminus X$ but not in $\mathbb{R}^2$.

Now: What set $X$ would you have to take to get $\dim(V/W) = 8$?

The SOLUTION OF THE 3D VERSION OF NASH’s PROBLEM can be obtained directly from the solution of the 2D version. How? This is a challenge problem.