Name:

MWF 9 Oliver Knill
MWF 9 Chao Li
MWF 10 Gijs Heuts
MWF 10 Yu-Wen Hsu
MWF 10 Yong-Suk Moon
MWF 11 Rosalie Belanger-Rioux
MWF 11 Gijs Heuts
MWF 11 Siu-Cheong Lau
MWF 12 Erick Knight
MWF 12 Kate Penner
TTH 10 Peter Smillie
TTH 10 Jeff Kuan
TTH 10 Yi Xie
TTH 11:30 Jeff Kuan
TTH 11:30 Jameel Al-Aidroos

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

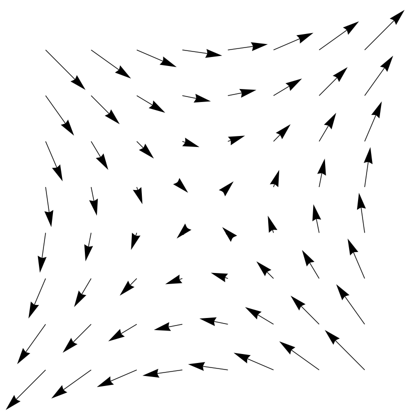
1)	T	F	The function $f(x,y,z) = x^2 - y^2 - z^2$ increases in the direction $\langle -3,-1,2\rangle/\sqrt{14}$ at the point $(1,1,1)$.
2)	T	F	The unit tangent vector of the curve $\vec{r}(t) = \langle 3t, 4t, t^2\rangle$ at time $t=0$ is $\langle 3/5, 4/5, 0\rangle$.
3)	T	F	There exist two nonzero vectors $\vec{a}$ and $\vec{b}$ such that the length of the vector projection of $\vec{a}$ to $\vec{a}\times\vec{b}$ is $\frac{1}{2}\lvert\vec{b}\rvert$.
4)	T	F	The arc length of the curve $\vec{r_1}(t) = \langle e^{3t^3}-1, t^6+2, \sin(2t^3)\rangle$, $0 \le t \le 1$ is larger than that of $\vec{r_2}(t) = \langle e^{3t}-1, t^2+2, \sin(2t)\rangle$, $0 \le t \le 1$.
5)	T	F	The tangent plane of the graph of $f(x,y) = \sin(x)+y^3$ at $(0,1,1)$ is $x+3y=3$.
6)	T	F	There exists a curve C on the level surface of $f(x,y,z) = x^3+e^{yz}+\cos(y) = 2$ such that the line integral $\int_C \nabla f \cdot d\vec{r} > 0$.
7)	T	F	If Q is the point away from the plane $3x+5y+z=7$ and P is the point on the plane closest to Q, then $\vec{PQ}$ is parallel to $\langle 3,5,1\rangle$.
8)	T	F	The vector field $\vec{F}(x,y,z) = \langle y^2 - xz + e^y, -yz, x^4+y^2-z^2\rangle$ is the curl of a vector field $\vec{G}$.
9)	T	F	Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle$ and C be the unit circle oriented counterclockwise. Since $Q_x = P_y$ everywhere, Green implies $\int_C \vec{F}\cdot d\vec{r} = 0$.
10)	T	F	By linear approximation of the function $f(x,y,z) = e^{x+y+z}$ we can estimate $f(0.1, 0.01, 0.001)$ as 1.111.
11)	T	F	If $\vec{F}(x,y,z)$ is a vector field defined on $0 < x^2+y^2+z^2 < 4$ and $\operatorname{curl}(\vec{F}) = 0$ everywhere on this solid, then $\vec{F} = \nabla f$ for some function f.
12)	T	F	The tangent plane of the surface $x^2+y^4+z^6 = 6$ at $(2,1,1)$ is perpendicular to the line $\vec{r}(t) = \langle 1+2t, 3+2t, -4+3t\rangle$.
13)	T	F	Given two curves $C_1 : \vec{r}_1(t) = \langle t, t^3\rangle, 0 \le t \le 1$ and $C_2 : \vec{r}_2(s) = \langle s, s^5\rangle, 0 \le s \le 1$ $f(x,y) = \sin(x^2 y)$. Then $\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$.
14)	T	F	If $f(x,y)$ has a global maximum, then the discriminant function $D(x,y) = f_{xx}f_{yy} - f_{xy}^2$ has a global maximum.
15)	T	F	Let $\vec{F}(x,y,z) = \langle x,y,z\rangle$ and S the surface boundary of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ oriented by outward normal vectors. Then $\int\int_S \vec{F}\cdot d\vec{S} = 0$.
16)	T	F	Let $\vec{F}(x,y,z) = \langle x/3, y/3, z/3\rangle$ and S the unit sphere oriented by the outward normal vectors. Then $\int\int_S \operatorname{curl}(\vec{F})\cdot d\vec{S}$ is the volume of the unit ball.
17)	T	F	In three dimensional space there exist two nonzero vector fields $\vec{F}$ and $\vec{G}$ such that $\operatorname{curl}(\vec{F}) = \operatorname{div}(\vec{G})$.
18)	T	F	The vector field $\vec{F}(x,y,z) = \langle \cos(y), \cos(z), \cos(x)\rangle$ has the property that $\vec{F} = \operatorname{curl}(\operatorname{curl}(\vec{F}))$.
19)	T	F	There exists a vector field $\vec{F}(x,y,z)$ defined on $\mathbf{R}^3$ such that every line integral $\int_C \vec{F}\cdot d\vec{r}$ of $\vec{F}$ over a closed curve C is equal to 0, but not every surface integral $\int\int_S \vec{F}\cdot dS$ over a closed surface S is equal to 0.
20)	T	F	Whenever $\vec{F} = \nabla f$, for some function $f(x,y)$ defined on the annulus $\frac{1}{2} \le x^2+y^2 \le 2$, then $\int_C \vec{F}\cdot d\vec{r} = 0$, where C is the circle $x^2+y^2=1$.

Problem 2) (10 points)

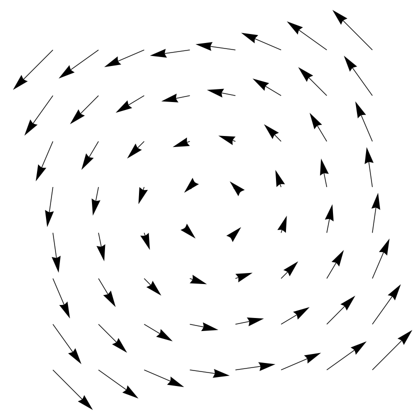
a) (5 points) We match in this problems vector fields with properties of vector fields and formulas for vector fields. A field $\vec{F}$ is **divergence free** if $\text{div}(\vec{F}) = 0$ everywhere in the plane. A field $\vec{F}$ is **irrotational**, if $\text{curl}(\vec{F}) = \vec{0}$ everywhere in the plane. In the last two columns of the following table, check the boxes which apply.

field	enter I-IV	divergence free	irrotational
$\vec{F}(x,y) = \langle -y, x \rangle$			
$\vec{F}(x,y) = \langle y, x \rangle$			
$\vec{F}(x,y) = \langle -x-y, x-y \rangle$			
$\vec{F}(x,y) = \langle x+y, x+y \rangle$			

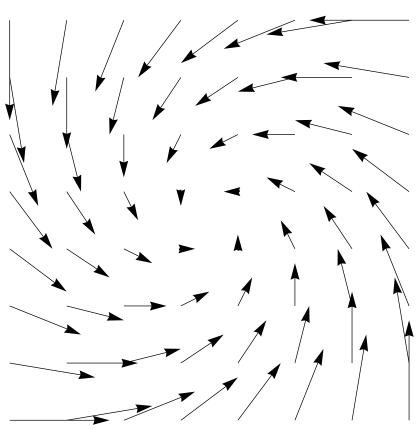
I



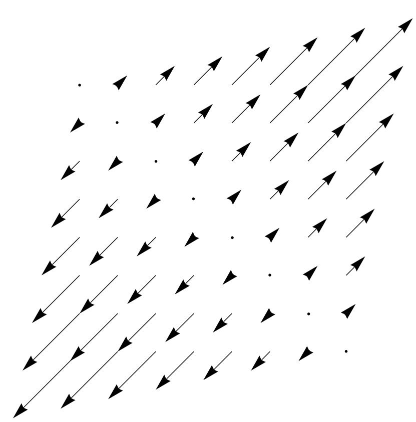
II



III



IV



b) (5 points) Match the following names of partial differential equations with functions $u(t,x)$ which satisfy the differential equation and with formulas defining these equations.

equation	A-D	1-4
wave		
heat		
transport		
Laplace		

A	$u(t,x) = t^2 + x^2$
B	$u(t,x) = t^2 - x^2$
C	$u(t,x) = \sin(x+t)$
D	$u(t,x) = x^2 + 2t$

1	$u_t(t,x) = u_x(t,x)$
2	$u_{tt}(t,x) = u_{xx}(t,x)$
3	$u_{tt}(t,x) = -u_{xx}(t,x)$
4	$u_t(t,x) = u_{xx}(t,x)$

Problem 3) (10 points)

a) (6 points) Select 6 of the integrals $A-H$ in the lower tables and match them with their names in the following table:

name	label A-H
line integral	
flux integral	
surface area	
arc length	
volume	
area	

$\int\int_R x^2 - y^2 \; dxdy$	A
$\int\int_R 1 \; dxdy$	B
$\int\int\int_R 1 \; dxdydz$	C
$\int\int\int_R x^2 + z^2 \; dxdydz$	D

$\int\int_R \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \; dudv$	E
$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \; dt$	F
$\int_a^b \lvert\vec{r}\,'(t)\rvert \; dt$	G
$\int\int_R \lvert\vec{r}_u \times \vec{r}_v\rvert \; dudv$	H

b) (4 points)

derivative	enter A-D
divergence	
curl	
gradient	
directional derivative	

The middle column of the following table is obtained by applying a derivative operation to the object in the left column. Fill in the correct label (A-D) of that operation into the above table.

object	derivative	label
$\vec{F}(x,y,z) = \langle -y, x, x\rangle$	$\langle 0, -1, 2\rangle$	A
$\vec{F}(x,y,z) = \langle x^2, y, x\rangle$	$2x + 1$	B
$f(x,y,z) = x^2 + y^2 + z$	$\langle 2x, 2y, 1\rangle$	C
$f(x,y,z) = x^3 + 5y^2$	$10y$	D

Problem 4) (10 points)

Consider the tetrahedron with vertices

$$A = (0, 1, -1), B = (4, 0, -1), C = (2, 1, 3), \text{ and } D = (2, 2, 0) .$$

a) (3 points) What is the area of the parallelogram spanned by $\vec{AB}$ and $\vec{AD}$?

b) (3 points) Find the volume of the parallelepiped spanned by $\vec{AC}$, $\vec{AB}$ and $\vec{AD}$.

c) (4 points) Determine the distance between the two skew lines AB and CD.

Problem 5) (10 points)

a) (5 points) The curl of $\vec{F}(x, y) = \langle -e^{xy}, y\rangle$ is equal to a scalar function $f(x, y)$. Estimate $f(1.1, 0.001)$ by linear approximation.

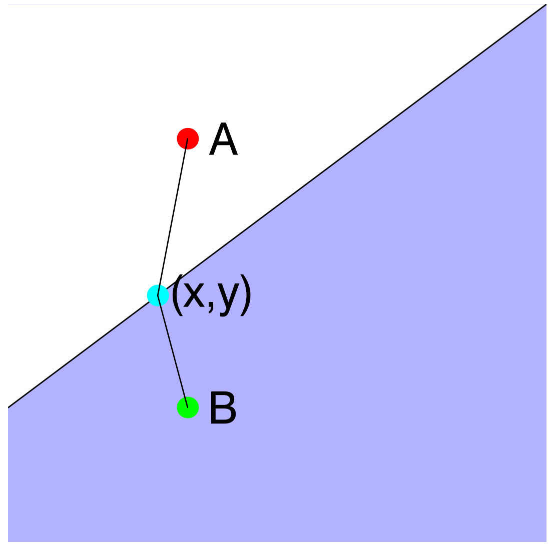
b) (5 points) Using the same function as in a), the equation $f(x, y) = \text{curl}(\vec{F})(x, y) = 1$ defines y as a function $g(x)$ of x near $x = 1$. Find $g'(1)$.

Problem 6) (10 points)

Find all the critical points of the function $f(x, y) = y^3 - 3y^2 + 4x + x^2 - 3$ and classify them by telling whether they are local maxima, local minima or saddle points.

Problem 7) (10 points)

A nightelf in the game World of Warcraft runs from $A = (0, 2)$ to $B = (0, 0)$ along a straight line segment from A to (x, y) and swims through the lake $x - y \geq -1$ from (x, y) to a gold chest located at $B = (0, 0)$ again on a straight line segment. The effort from A to (x, y) is the square of the distance from A to (x, y). Her effort from (x, y) to B is 2 times the squared distance from (x, y) to B. Using the Lagrange method, find the choice of a drop point (x, y) on the lake shore that minimizes her effort.



Problem 8) (10 points)

Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r} \,,$$

where C is the curve given by

$$\vec{r}(t) = \langle \frac{t\pi}{2}, 1-t, t^3 \rangle, 0 \leq t \leq 1$$

and

$$\vec{F}(x,y,z) = \langle e^{y^2} + z\cos(xz), 2xye^{y^2}, x\cos(xz) \rangle \ .$$

Problem (9) (10 points)

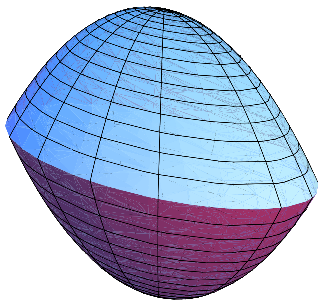
The picture shows an unidentified flying object (UFO). Although it is unidentified, we know its shape. One part of the surface

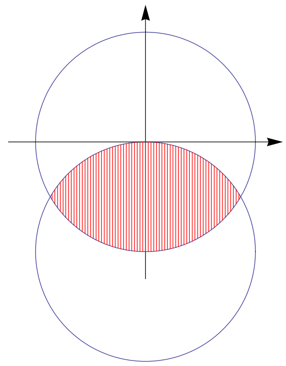
$$x^2 + y^2 + z^2 = 4$$

and the other part of the surface is

$$x^2 + y^2 + (z+2)^2 = 4 \ .$$

Find the surface area of the UFO.





Problem 10) (10 points)

Evaluate the following integral

$$\int_0^2 \int_1^3 \int_{z^2}^4 xz\cos(y^2)\; dydxdz\ .$$

Problem 11) (10 points)

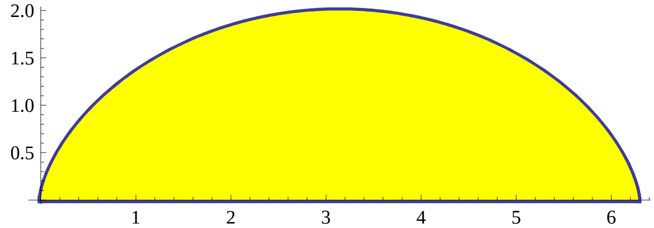
Let $\vec{F}(x,y,z) = \langle x + yz, xye^{-xz}, e^{-xz}\rangle$. Find

$$\int\int_S \vec{F} \cdot d\vec{S}\ ,$$

where S is the surface $z = 1 - x^2 - y^2, z \geq 0$ oriented so that the normal vector points upwards.

Problem 12) (10 points)

Find the area of the region on the plane enclosed by the curve $\vec{r}(t) = \langle t - \sin(t), 1 - \cos(t)\rangle$ with $0 \leq t \leq 2\pi$ and the x-axes.



Problem 13) (10 points)

Evaluate the integral

$$\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S} \ ,$$

where $\vec{F}(x, y, z) = \langle xe^{y^2}z^3 + 2xyze^{x^2+z}, x + z^2e^{x^2+z}, ye^{x^2+z} + ze^x \rangle$ and where S is the part of the ellipsoid $x^2 + y^2/4 + (z + 1)^2 = 2$, $z > 0$ oriented so that the normal vector points upwards.

Problem 14) (10 points)

Let E be the rectangular solid $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq 1$ and let S be the boundary of E. The surface S consists of 6 planar pieces where each is oriented so that the normal vector points outwards. Given the vector field

$$\vec{F} = \langle -x^2 - 4xy, -yz, 12z \rangle \ ,$$

for which parameters a, b is the flux integral

$$\int\int_S \vec{F} \cdot d\vec{S}$$

a global maximum?