| Name: |
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| MWF 9 Oliver Knill |
| MWF 9 Chao Li |
| MWF 10 Gijs Heuts |
| MWF 10 Yu-Wen Hsu |
| MWF 10 Yong-Suk Moon |
| MWF 11 Rosalie Belanger-Rioux |
| MWF 11 Gijs Heuts |
| MWF 11 Siu-Cheong Lau |
| MWF 12 Erick Knight |
| MWF 12 Kate Penner |
| TTH 10 Peter Smillie |
| TTH 10 Jeff Kuan |
| TTH 10 Yi Xie |
| TTH 11:30 Jeff Kuan |
| TTH 11:30 Jameel Al-Aidroos |

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

| | | |
|---|---|---|
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| 11 | | 10 |
| 12 | | 10 |
| 13 | | 10 |
| 14 | | 10 |
| Total: | | 150 |

Problem 1) True/False questions (20 points). No justifications are needed.

1) T F There are two unit vectors $\vec{v}$, $\vec{w}$ for which the sum $\vec{v}+\vec{w}$ has length 1/3.

2) T F For any three vectors, we have $|(\vec{u} \times \vec{v}) \times \vec{w}| = |(\vec{v} \times \vec{w}) \times \vec{u}|$.

3) T F Denote by $d(P, L)$ the distance from a point $P$ to a line $L$ in space. For any point $P$ and any two lines $L, K$ in space, we have $d(P, L) + d(P, K) \geq d(L, K)$.

4) T F For any three vectors $\vec{u}, \vec{v}, \vec{w}$, the relation $|\vec{u} \times (\vec{v} \times \vec{w})| \leq |\vec{u}||\vec{v}||\vec{w}|$ holds.

5) T F If $\vec{r}(t)$ has speed 1 and curvature 1 everywhere, then $\vec{r}(2t)$ has constant speed 2 and constant curvature 1/2 everywhere.

6) T F If the curvature of a space curve is constant 1 and the speed $|\vec{r}\,'(t)| = 1$ everywhere, then the acceleration satisfies $|\vec{r}\,''(t)| = 1$ everywhere.

7) T F If a vector field $\vec{F} = \langle P, Q \rangle$ has $\mathrm{curl}(\vec{F}) = Q_x - P_y = 0$ everywhere and divergence $\mathrm{div}(\vec{F}) = P_x + Q_y = 0$ everywhere, then $\vec{F}$ must be constant.

8) T F If the level curve $f(x, y) = 1$ contains both the lines $x = y$ and $x = -y$, then $(0, 0)$ must be a critical point for which $D < 0$.

9) T F The surface $\vec{r}(u, v) = \langle u^3 \cos(v), u^3 \sin(v), u^3 \rangle$ with $v \in [0, 2\pi)$ and $-\infty \leq u \leq \infty$ is a double cone.

10) T F There is a non-constant function $f(x, y, z)$ of three variables such that $\mathrm{div}(\mathrm{grad}(f)) = f$.

11) T F If $\mathrm{curl}(\vec{F}) = \vec{F}$, then the vector field $\vec{F}$ satisfies $\mathrm{div}(\vec{F}) = 0$ everywhere.

12) T F The equation $\phi = \pi/4$ in spherical coordinates defines a half plane.

13) T F The tangent plane of $x^3 + y^2 + z^4 = 9$ at $(0, 3, 0)$ is $y = 3$.

14) T F Assume $(x_0, y_0)$ is not a critical point of $f(x, y)$. It is possible that $f$ increases at $(x_0, y_0)$ most rapidly in the direction $\langle 1, 0 \rangle$ and decreases most rapidly in the direction $\langle 4/5, -3/5 \rangle$.

15) T F Assume $\vec{F}(x, y, z)$ is defined everywhere except on the $z$-axis and satisfies $\mathrm{curl}(\vec{F}) = \vec{0}$ everywhere except on the $z$-axis, then $\int_C \vec{F} \cdot d\vec{r} = 0$ for all curves $C$.

16) T F A point $(x_0, y_0)$ is an extremum of $f(x, y)$ under the constraint $g(x, y) = 0$. If $D = f_{xx} f_{yy} - f_{xy}^2 > 0$, then $(x_0, y_0)$ can not be a local maximum on the constraint curve.

17) T F The vector field $\vec{F}(x, y, z) = \langle x^2, y^2, z^2 \rangle$ can be the curl of another vector field $\vec{G}$.

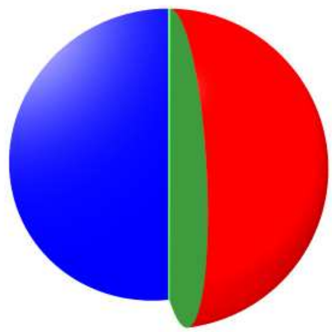
18) T F If $f(x, y)$ and $g(x, y)$ are two functions and $(2, 3, 3)$ is a critical point of the function $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$, then $(2, 3)$ is a solution of the Lagrange equations for extremizing $f(x, y)$ under the constraint $g(x, y) = 0$.

19) T F Assume $(0, 0)$ is a global maximum of $f(x, y)$ on the disc $D = \{x^2 + y^2 \leq 1\}$, then $\int\int_D f(x, y)\, dxdy \leq \pi f(0, 0)$.

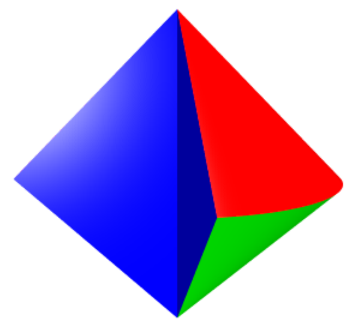
20) T F Let $C$ be a curve parametrized by $\vec{r}(t), 0 \leq t \leq 1$ for which the acceleration is constant 1. Then $\int_C \nabla f \cdot d\vec{r}$ is equal to $\int_0^1 D_{\vec{r}\,''(t)}(f(\vec{r}(t)))\, dt$.

Problem 2) (10 points)

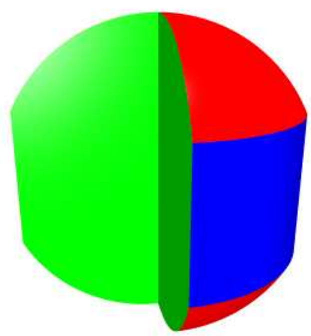
a) (4 points) Match the following triple integrals with the regions.



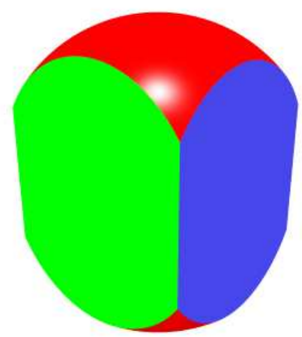
I



II



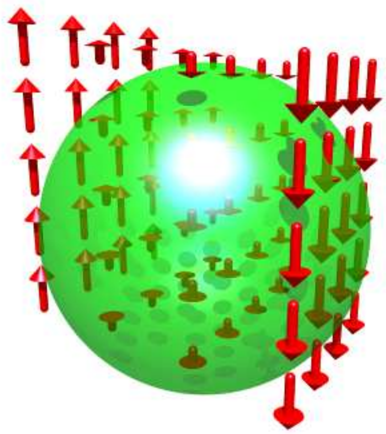
III



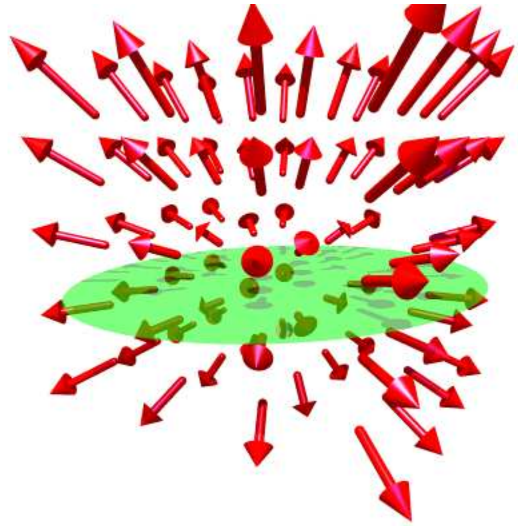
IV

| Enter I,II,III,IV here | Equation |
|---|---|
| | $\int_0^{3\pi/2} \int_0^1 \int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} f(r\cos(\theta), r\sin(\theta), z) r \; dzdrd\theta$ |
| | $\int_0^{3\pi/2} \int_0^1 \int_{r-1}^{1-r} f(r\cos(\theta), r\sin(\theta), z) r \; dzdrd\theta$ |
| | $\int_0^{3\pi/2} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} f(r\cos(\theta), r\sin(\theta), z) r \; dzdrd\theta$ |
| | $\int_{-1}^1 \int_{-1}^1 \int_{-\sqrt{2-x^2-y^2}}^{\sqrt{2-x^2-y^2}} f(x, y, z) \; dzdydx$ |

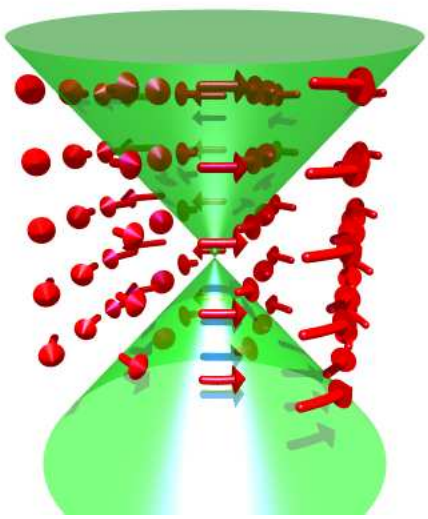
2b) (6 points) Match the following pictures with their vector fields and surfaces. Then check whether the flux integral is zero.



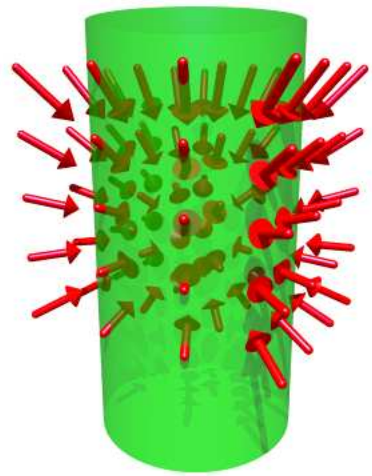
A



B



C

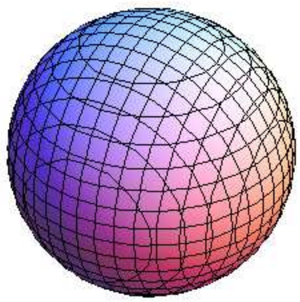


D

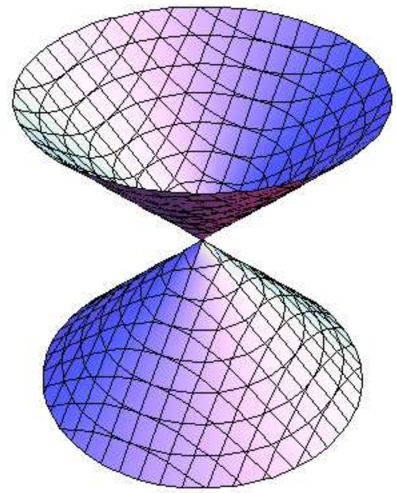
| Enter A,B,C,D | Field | Surface | Flux zer |
|---|---|---|---|
| | $\vec{F}(x,y,z) = \langle x, y, z \rangle$ | $\vec{r}(u,v) = \langle u, v, 0 \rangle$ | |
| | $\vec{F}(x,y,z) = \langle 0, 0, y \rangle$ | $\vec{r}(\theta,\phi) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi) \rangle$ | |
| | $\vec{F}(x,y,z) = \langle -x, -y, -z \rangle$ | $\vec{r}(\theta,z) = \langle \cos(\theta), \sin(\theta), z \rangle$ | |
| | $\vec{F}(x,y,z) = \langle -y, x, 0 \rangle$ | $\vec{r}(\theta,z) = \langle z\cos(\theta), z\sin(\theta), z \rangle$ | |

Problem 3) (10 points)

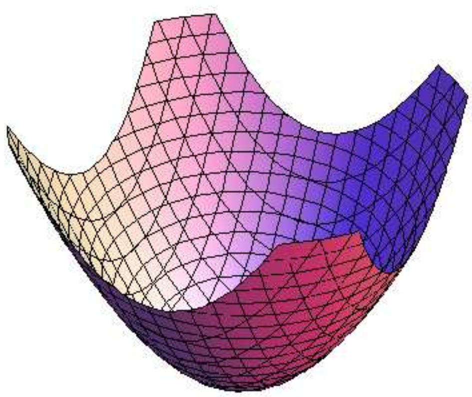
a) (6 points) Match the following level surfaces with functions $f(x,y,z)$ and also match the parametrization of part of the surface $f(x,y,z)=0$.



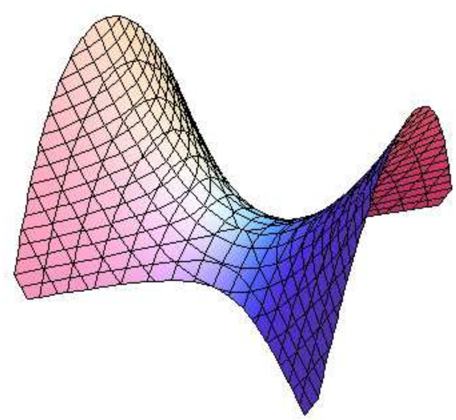
I



II



III



IV

| Enter I,II,III,IV | $f(x,y,z)=0$ |
|---|---|
| | $f(x,y,z)=-x^2+y^2+z$ |
| | $f(x,y,z)=x^2+y^2+z^2-1$ |
| | $f(x,y,z)=-x^2-y^2+z$ |
| | $f(x,y,z)=-x^2-y^2+z^2$ |

| Enter I,II,III,IV | parametrization |
|---|---|
| | $\langle u,v,u^2-v^2\rangle$ |
| | $\langle u,v,u^2+v^2\rangle$ |
| | $\langle u,v,\sqrt{1-u^2-v^2}\rangle$ |
| | $\langle s\cos(t),s\sin(t),s\rangle$ |

b) (2 points) We know that $\vec{r}\,''(t) = \langle -\cos(t), -\sin(t), 0 \rangle$, $\vec{r}(0) = \langle 2, 3, 4 \rangle$ and $\vec{r}\,'(0) = \langle 0, 1, 1 \rangle$. The expression $\langle \cos(t)+1, \sin(t)+3, t+4 \rangle$ is equal to:

| Check which applies | result |
|---|---|
| | the velocity $\vec{r}\,'(t)$ |
| | the position $\vec{r}(t)$ |
| | the curvature $\kappa(\vec{r}(t))$ |
| | the unit tangent vector $\vec{T}(t)$ |

c) (2 points) What is the name of the partial differential equation $\text{div}(\text{grad}(f)) = 0$ for $f(x, y)$?

| Check which applies | PDE |
|---|---|
| | Transport equation |
| | Wave equation |
| | Heat equation |
| | Laplace equation |

## Problem 4) (10 points)

Find the distance between the sphere $(x-4)^2 + y^2 + (z-6)^2 = 1$ and the cylinder of radius 2 around the line $x = y = z$.

## Problem 5) (10 points)

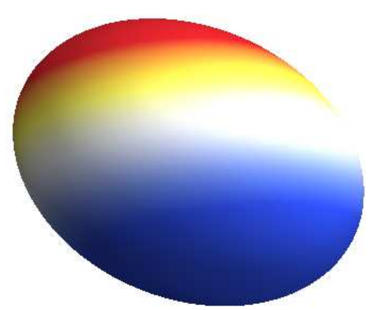
a) (3 points) Find the tangent plane to the surface $S : 4xy - z^2 = 0$ at $(1, 1, 2)$.

b) (4 points) Estimate $4 * 1.001 * 0.99 - 2.001^2$, where $*$ is the usual multiplication.

c) (3 points) Parametrize the line through $(1, 1, 2)$ which is perpendicular to the surface $S$ at $(1, 1, 2)$.

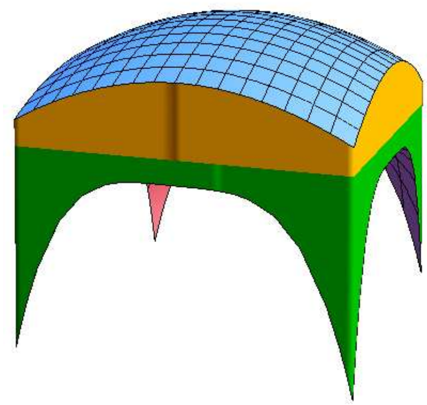
## Problem 6) (10 points)

Find the place on the elliptical **asteroid** surface $g(x, y, z) = 5x^2 + y^2 + 3z^2 = 9$, where the temperature $f(x, y, z) = 750 + 5x - 2y + 9z$ is maximal.



| Problem 7) (10 points) |
|---|

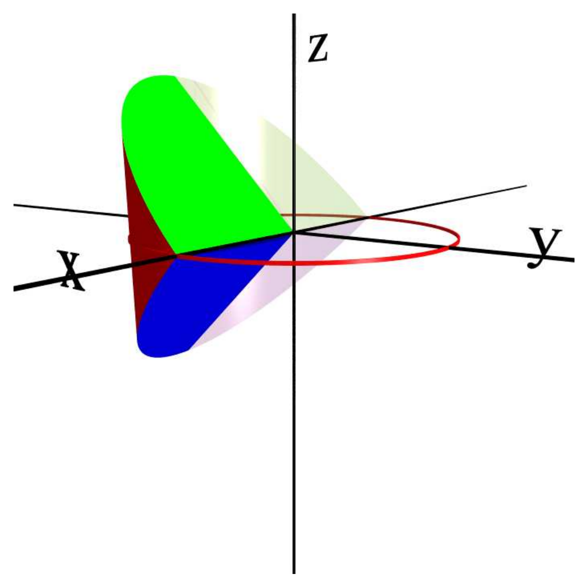
The thickness of the region enclosed by the two graphs $f_1(x,y) = 10 - 2x^2 - 2y^2$ and $f_2(x,y) = -x^4 - y^4 - 2$ is denoted by $f(x,y) = f_1(x,y) - f_2(x,y)$. Classify all critical points of $f$ and find the global minimal thickness.



To the picture: over a square domain, the region looks like a chair. In the problem you consider the function over the entire plane.

| Problem 8) (10 points) |
|---|

Find the volume of the solid piece of **cheese** bound by the cylinder $x^2 + y^2 = 1$, the planes $y - z = 0$ (bottom boundary) and $y + z = 0$ (top boundary) which is on the quadrant $x \geq 0$ and $y \leq 0$.




| Problem 9) (10 points) |
|---|

Compute the surface area of the **Tsai** surface which is parametrized by

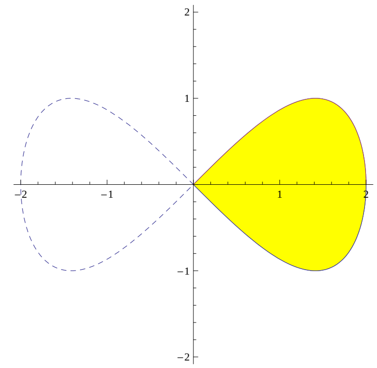
$$\vec{r}(u,v) = \langle 3u + 2v, 4u + v, \frac{2}{7}v^{\frac{7}{2}} \rangle ,$$

where $0 \leq u \leq 1$ and $u^{1/4} \leq v \leq 1$.

Problem 10) (10 points)

Find the area $\int\int_R 1 \; dxdy$ of the region $R$ inside the right leaf of the **Gerono lemniscate** $x^4 = 4(x^2 - y^2)$ which has the parametrization

$$\vec{r}(t) = \langle 2\sin(t), 2\sin(t)\cos(t) \rangle \; .$$




Problem 11) (10 points)

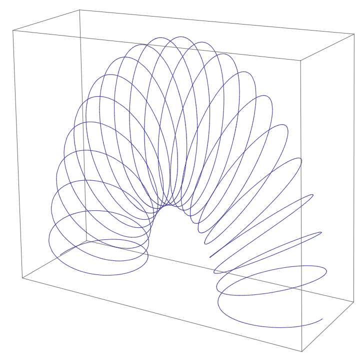
Find the line integral of the vector field

$\vec{F}(x,y,z) = \langle \cos(x+z), 2yze^{y^2z} + 7, \cos(x+z) + y^2e^{y^2z} \rangle$

along the **slinky** curve

$\vec{r}(t) = \langle \sin(40t), (2+\cos(40t))\cos(t), (2+\cos(40t))\sin(t) \rangle$

with $0 \leq t \leq \pi$.



Problem 12) (10 points)

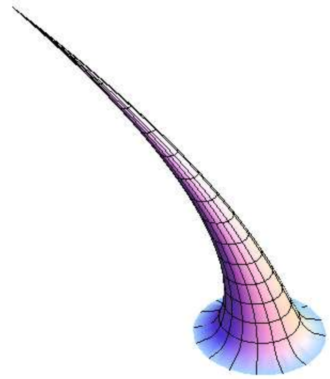
Find the flux integral $\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S}$, where

$$\vec{F}(x,y,z) = \langle 2\cos(\pi y)e^{2x} + z^2, x^2\cos(z\pi/2) - \pi\sin(\pi y)e^{2x}, 2xz \rangle$$

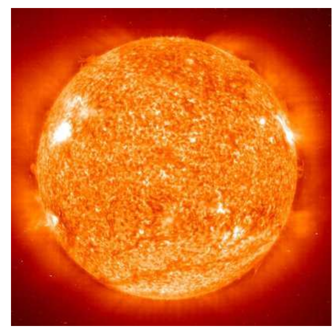
and $S$ is the **thorn** surface parametrized by

$$\vec{r}(s,t) = \langle (1 - s^{1/3})\cos(t) - 4s^2, (1 - s^{1/3})\sin(t), 5s \rangle$$

with $0 \leq t \leq 2\pi, 0 \leq s \leq 1$ and oriented so that the normal vectors point to the outside of the thorn.



Problem 13) (10 points)

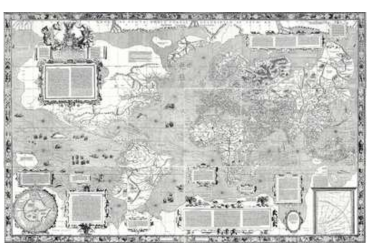


Assume the vector field

$\vec{F}(x, y, z) = \langle 5x^3 + 12xy^2, y^3 + e^y \sin(z), 5z^3 + e^y \cos(z) \rangle$

is the magnetic field of the **sun** whose surface is a sphere of radius 3 oriented with the outward orientation. Compute the magnetic flux $\int\int_S \vec{F} \cdot d\vec{S}$.

Problem 14) (10 points)



The **Mercator projection** is one of the most famous map projections. It was invented in 1569 and used for nautical voyages. The inverse of the projection is the parametrization of the sphere as

$$\vec{r}(u, v) = \langle \cos(u) \cos(\arctan(\sinh(v))), \sin(u) \cos(\arctan(\sinh(v))), \sin(\arctan(\sinh(v))) \rangle \ .$$

a) (3 points) Show that $|\vec{r}(u, v)| = 1$ verifying so that $\vec{r}(u, v)$ parametrizes the unit sphere, if $0 \leq u < 2\pi, -\infty < v < \infty$.
b) (3 points) Show that $|\vec{r}_u(u, v)| = |\vec{r}_v(u, v)| = 1/\cosh(v)$ and that $\vec{r}_u(u, v) \cdot \vec{r}_v(u, v) = 0$.
c) (2 points) Use *b*) to show that $|\vec{r}_u \times \vec{r}_v| = 1/\cosh(x)^2$.
d) (2 points) Use $\int 1/\cosh^2(x)\ dx = 2 \arctan(\tanh(x/2)) + C$ to see that the surface area of the unit sphere is $4\pi$.

Hint for b): you can use the identity $\cos(\arctan(\sinh(v) = 1/\cosh(v)$.