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MWF 11 Siu-Cheong Lau
MWF 12 Erick Knight
MWF 12 Kate Penner
TTH 10 Peter Smillie
TTH 10 Jeff Kuan
TTH 10 Yi Xie
TTH 11:30 Jeff Kuan
TTH 11:30 Jameel Al-Aidroos

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

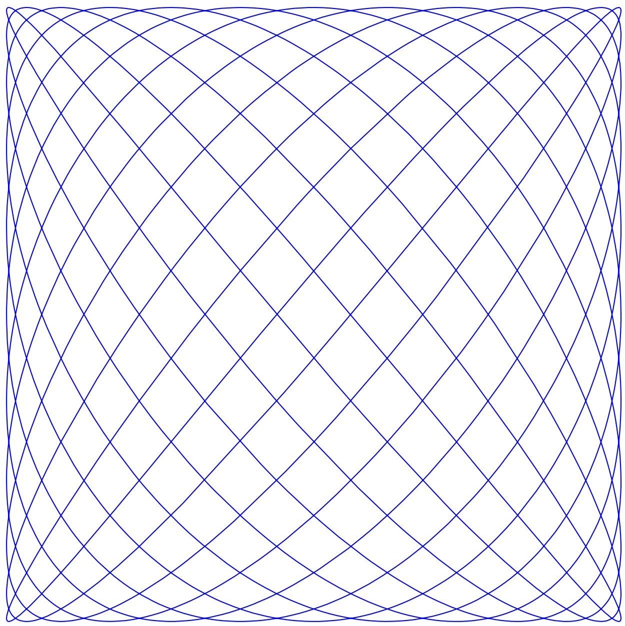
Problem 1) True/False questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

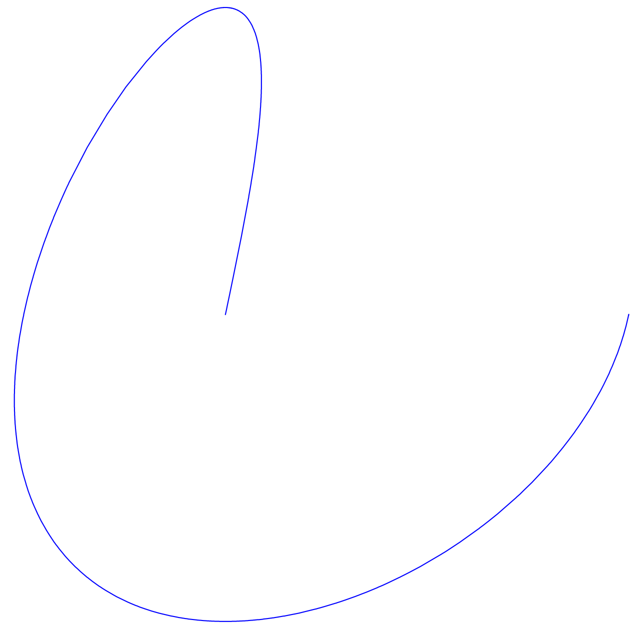
1)	T	F	The length of the curve $\vec{r}(t) = \langle \sin(t), t^4 + t, \cos(t) \rangle$ on $t \in [0, 1]$ is the same as the length of the curve $\vec{r}(t) = \langle \sin(t^2), t^8 + t^2, \cos(t^2) \rangle$ on $[0, 1]$.
2)	T	F	The parametric surface $\vec{r}(u, v) = (5u - 3v, u - v - 1, 5u - v - 7)$ is a plane.
3)	T	F	Any function $u(x, y)$ that obeys the differential equation $u_{xx} + u_x - u_y = 1$ has no local maxima.
4)	T	F	The length of the vector projection of $\vec{b}$ onto a vector $\vec{a}$ is smaller or equal than the length of the vector $\vec{b}$.
5)	T	F	If $f(x, y)$ is a function such that $f_x - f_y = 0$ then f is called conservative.
6)	T	F	$(\vec{u} \times \vec{v}) \cdot \vec{w} = (\vec{u} \times \vec{w}) \cdot \vec{v}$ for all vectors $\vec{u}, \vec{v}, \vec{w}$ in space.
7)	T	F	The equation $\rho = \phi/4$ in spherical coordinates is half a cone.
8)	T	F	If $f(x, y) = \frac{x^3}{x^2+y^2}$ and $(x, y) \to (0, 0)$ then $f(x, y) \to 0$.
9)	T	F	$\int_0^1 \int_0^x 1 \, dydx = 1/2$.
10)	T	F	Let $\vec{a}$ and $\vec{b}$ be two vectors which are perpendicular to a given plane Σ. Then $\vec{a} + \vec{b}$ is also perpendicular to Σ.
11)	T	F	If $g(x, t) = f(x - vt)$ for some function f of one variable $f(z)$ then g satisfies the differential equation $g_{tt} - v^2 g_{xx} = 0$.
12)	T	F	If $f(x, y)$ is a continuous function on $\mathbf{R}^2$ such that $\int\int_D f \, dA \geq 0$ for any region D then $f(x, y) \geq 0$ for all (x, y).
13)	T	F	Assume the two functions $f(x, y)$ and $g(x, y)$ have both the critical point $(0, 0)$ which are saddle points, then $f + g$ has a saddle point at $(0, 0)$.
14)	T	F	If $f(x, y)$ is a function of two variables and if $h(x, y) = f(g(y), g(x))$, then $h_x(x, y) = f_y(g(y), g(x))g'(y)$.
15)	T	F	If we rotate a line around the z axis, we obtain a cylinder.
16)	T	F	If $u(x, y)$ satisfies the transport equation $u_x = u_y$ everywhere in the plane, then the vector field $\vec{F}(x, y) = \langle u(x, y), u(x, y) \rangle$ is a gradient field.
17)	T	F	$3 \, \text{grad}(f) = \frac{d}{dt} f(x + t, y + t, z + t)$.
18)	T	F	If $\vec{F}$ is a vector field in space and f is equal to the line integral of $\vec{F}$ along the straight line C from $(0, 0, 0)$ to (x, y, z), then $\nabla f = \vec{F}$.
19)	T	F	The line integral of $\vec{F}(x, y) = (x, y)$ along an ellipse $x^2 + 2y^2 = 1$ is zero.
20)	T	F	The identity $\text{div}(\text{grad}(f)) = 0$ is always true.

Problem 2) (10 points)

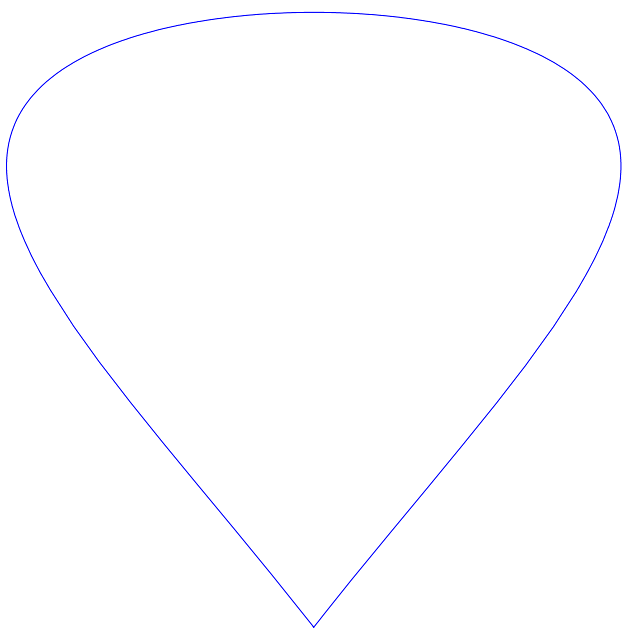
a) Match the equations with the curves. No justifications are needed.



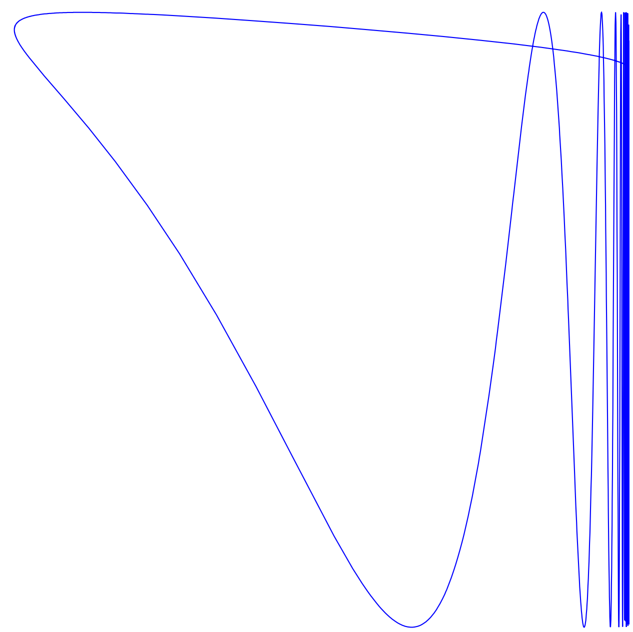
I



II



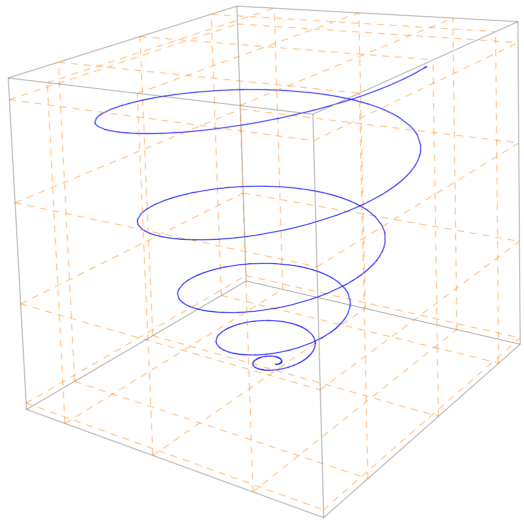
III



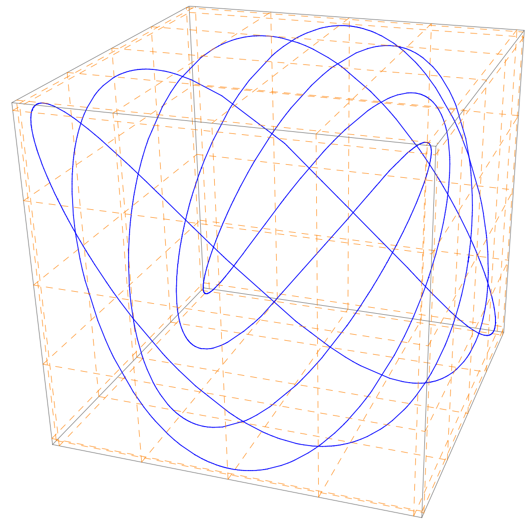
IV

Enter I,II,III,IV here	Equation
	$\vec{r}(t) = \langle \sin(t), t(2\pi - t) \rangle$
	$\vec{r}(t) = \langle \cos(11t), \sin(13t) \rangle$
	$\vec{r}(t) = \langle t\cos(t), \sin(t) \rangle$
	$\vec{r}(t) = \langle \cos(t), \sin(6/t) \rangle$

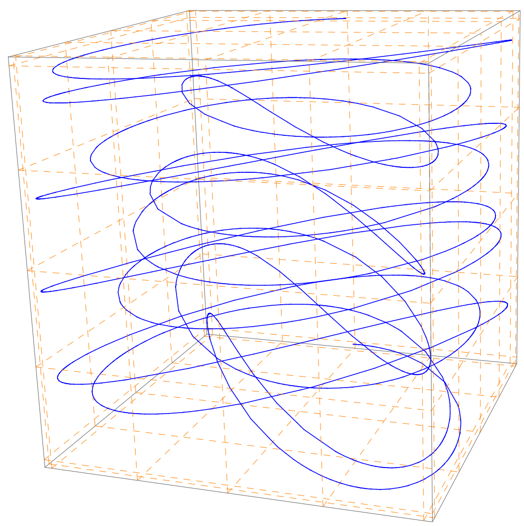
b) Match the parametrizations with the space curves. No justifications are needed.



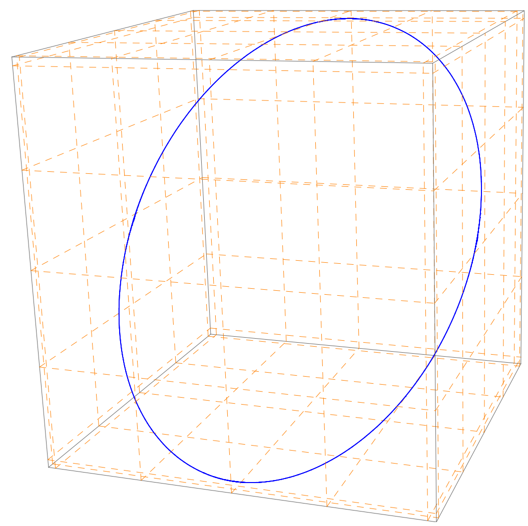
A



B



C



D

Enter A),B),C),D) here	Equation
	$\vec{r}(t) = \langle \cos(3t), \sin(3t), \sin(3t) \rangle$
	$\vec{r}(t) = \langle t\cos(t), t\sin(t), t^2 \rangle$
	$\vec{r}(t) = \langle \cos(5t), \sin(3t), \sin(7t) \rangle$
	$\vec{r}(t) = \langle \cos(13t), \sin(17t), t \rangle$

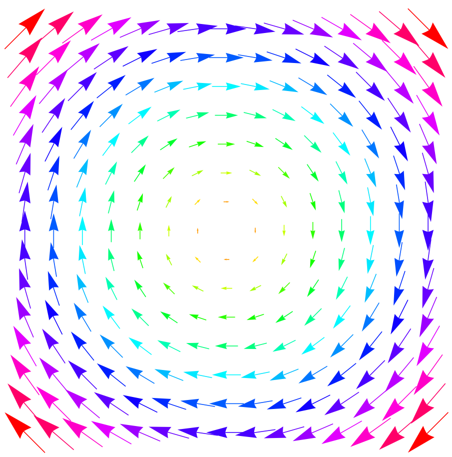
Problem 3) (10 points)

In this problem, vector fields $\vec{F}$ are written as $\vec{F} = \langle P, Q \rangle$. We use abbreviations $\mathrm{curl}(\vec{F}) = Q_x - P_y$ and $\mathrm{div}(\vec{F}) = P_x + Q_y$. When stating $\mathrm{curl}(\vec{F})(x, y) = 0$ we mean that $\mathrm{curl}(\vec{F})(x, y) = 0$ vanishes for **all** (x, y). The statement $\mathrm{curl}(\vec{F}) \neq 0$ means that $\mathrm{curl}(\vec{F})(x, y)$ does not

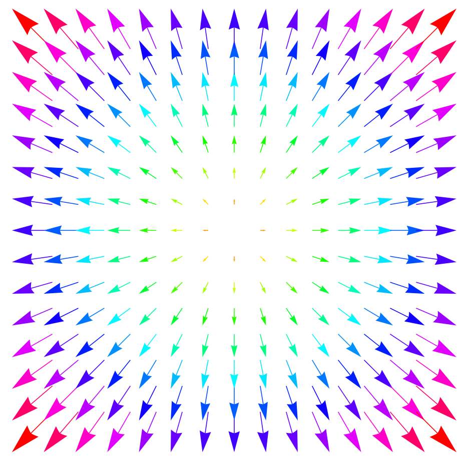
vanish for at least one point (x, y).
The same notation applies if curl is replaced by div. Check the box which match the formulas of the vector fields with the corresponding picture I,II,III or IV. Check also the boxes where curl and div are zero. In each of the four rows, you will need to check three boxes. No justifications are needed.

Vectorfield	I	II	III	IV	$\text{curl}(F) = 0$	$\text{curl}(F) \neq 0$	$\text{div}(F) = 0$	$\text{div}(F) \neq 0$
$\vec{F}(x, y) = \langle 0, 5 \rangle$								
$\vec{F}(x, y) = \langle y, -x \rangle$								
$\vec{F}(x, y) = \langle x, y \rangle$								
$\vec{F}(x, y) = \langle 2, x \rangle$								

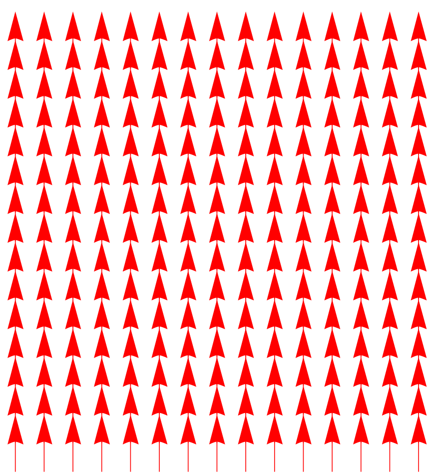
I



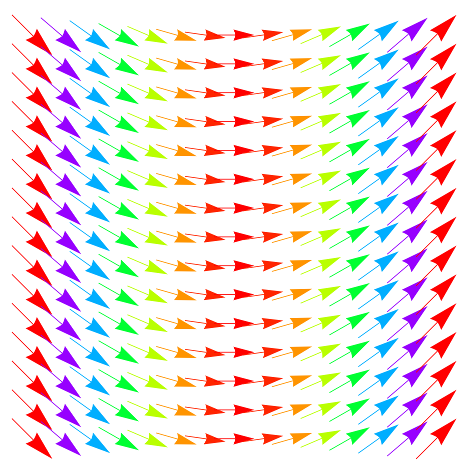
II



III



IV



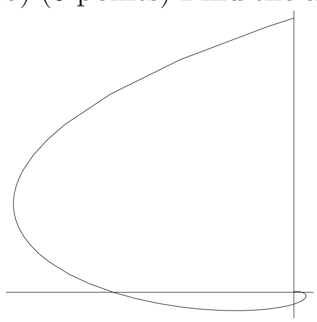
Problem 4) (10 points)

a) Find the scalar projection of the vector $\vec{v} = (3, 4, 5)$ onto the vector $\vec{w} = (2, 2, 1)$.

b) Find the equation of a plane which contains the vectors $\langle 1, 1, 0 \rangle$ and $\langle 0, 1, 1 \rangle$ and contains the point $(0, 1, 0)$.

Problem 5) (10 points)

a) (5 points) Find the surface area of the ellipse cut from the plane $z = 2x + 2y + 1$ by the cylinder $x^2 + y^2 = 1$.
b) (5 points) Find the arc length of the plane curve $\vec{r}(t) = (\sin(t)e^t, \cos(t)e^t)$ for $t \in [0, 2\pi]$.



Problem 6) (10 points)

a) Verify that if $u(x, y)$ and $v(x, y)$ are two functions, then $(uv)_{xx} = u_{xx}v + 2u_x v_x + uv_{xx}$.
b) The identity $\Delta(uv) = (\Delta u)v + u(\Delta v) + 2\nabla u \cdot \nabla v$ holds.
c) Assume u and v satisfy the Laplace equation $\Delta u = u_{xx} + u_{yy} = 0$ and $\nabla u \cdot \nabla v = 0$ then uv satisfies the Laplace equation.

Problem 7) (10 points)

Let $f(x, y, z) = 2x^2 + 3xy + 2y^2 + z^2$ and let R denote the region in space, where $2x^2 + 2y^2 + z^2 \leq 1$. Find the maximum and minimum values of f on the region R and list all points, where the maximum and minimum values are achieved. Distinguish between local extrema in the interior and extrema on the boundary.

Problem 8) (10 points)

Sketch the region of integration of the following iterated integral and then evaluate the integral:

$$\int_0^{\pi} \left(\int_{\sqrt{z}}^{\sqrt{\pi}} \left(\int_0^{x} \sin(xy) dy \right) dx \right) dz \ .$$

Problem 9) (10 points)

Evaluate the line integral

$$\int_C \vec{F} \cdot \vec{dr} \ ,$$

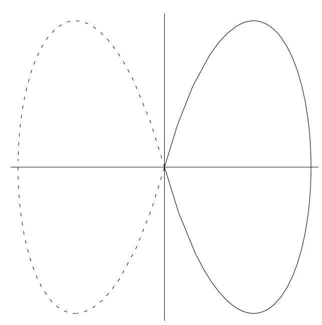
where C is the planar curve $\vec{r}(t) = \langle t^2, t/\sqrt{t+2} \rangle$, $t \in [0, 2]$ and $\vec{F}$ is the vector field $\vec{F}(x, y) = \langle 2xy, x^2 + y \rangle$. Do this in two different ways:

a) by verifying that $\vec{F}$ is conservative and replacing the path with a different path connecting $(0, 0)$ with $(4, 1)$,

b) by finding a potential function $f(x, y)$ which satisfies $\nabla f(x, y) = \vec{F}(x, y)$.

Problem 10) (10 points)

Evaluate the line integral $\int_C \vec{F} \cdot \vec{dr}$, where $\vec{F} = (x + e^x \sin(y), x + e^x \cos(y))$ and C is the right handed loop of the lemniscate described in polar coordinates as $r^2 = \cos(2\theta)$. The lemniscate part on the positive x half-plane is oriented counterclockwise.



Problem 11) (10 points)

Let $f(x, y, z)$ be the distance to the surface $x^4 + 2y^4 + z^4 = 1$. Show that f is a solution of the partial differential equation

$$f_x^2 + f_y^2 + f_z^2 = 1$$

outside the curve.

Hint: no computations are needed here. The shape of the surface pretty much irrelevant. What does the PDE say about the gradient ∇f?

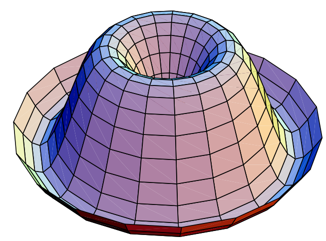
Problem 12) (10 points)

a) Find the line integral $\int_C \vec{F} \cdot d\vec{r}$ of the vector field $\vec{F}(x, y) = (xy, x)$ along the unit circle $C : t \mapsto \vec{r}(t) = (\cos(t), \sin(t)), t \in [0, 2\pi]$ by doing the actual line integral.

b) Find the value of the line integral obtained in part a) by evaluating a double integral.

Problem 13) (10 points)

Consider the surface given by the graph of the function $z = f(x, y) = \frac{100}{1+x^2+y^2} \sin\left(\frac{\pi}{8}(x^2 + y^2)\right)$ in the region $x^2 + y^2 \leq 16$. The surface is pictured to the right.



A magnetic field $\vec{B}$ is given by the curl of a vector potential $\vec{A}$. That is, $\vec{B} = \nabla \times \vec{A} = \text{curl}(\vec{A})$ and $\vec{A}$ is a vector field too. Suppose

$$\vec{A} = \langle z \sin(x^3), x\left(1 - z^2\right), \log(1 + e^{x+y+z})\rangle \ .$$

Compute the flux of the magnetic field through this surface. The surface has an upward pointing normal vector.

Problem 14) (10 points)

Let S be the surface given by the equations $z = x^2 - y^2$, $x^2 + y^2 \leq 4$, with the upward pointing normal. If the vector field $\vec{F}$ is given by the formula $\vec{F}(x, y, z) = \langle -x, y, \sqrt{x^2 + y^2}\rangle$, find the flux of $\vec{F}$ through S.