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- Print your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) True/False questions (20 points). No justifications are needed.

1) T F The parametrization $\vec{r}(u,v) = \langle \cos(u), \sin(u), v \rangle$ describes a cone.

2) T F The vectors $\vec{v} = \langle 2,1,5 \rangle$ and $\vec{w} = \langle 2,1,-1 \rangle$ are perpendicular.

3) T F Let E be a solid region with boundary surface S. If $\iint_S \vec{F} \cdot d\vec{S} = 0$, then $\text{div}(\vec{F})(x,y,z) = 0$ everywhere inside E.

4) T F If $\text{div}(\vec{F})(x,y,z) = 0$ for all (x,y,z) then $\iint_S \vec{F} \cdot d\vec{S} = 0$ for any closed surface S.

5) T F If $\vec{F}$ is a conservative vector field in space, then $\vec{F}$ is has zero divergence everywhere.

6) T F If $\vec{F}, \vec{G}$ are two vector fields for which $\vec{F} - \vec{G} = \text{curl}(\vec{H})$, then $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{G} \cdot d\vec{S}$ for any closed surface S.

7) T F The linearization of $f(x,y) = x^2 + y^3 - x$ at $(2,1)$ is $L(x,y) = 3 + 4(x-2) + 3(y-1)$.

8) T F The volume of the solid E is $\int\int_S \langle x, 2x+z, x-y \rangle \cdot d\vec{S}$, where S is the surface of the solid E.

9) T F The vector field $\vec{F}(x,y,z) = \langle x+y, x-y, 3 \rangle$ has zero curl and zero divergence everywhere.

10) T F If $\vec{F}$ is a vector field, then the flux of the vector field curl(curl(F)) through a sphere $x^2 + y^2 + z^2 = 1$ is zero.

11) T F If the vector field $\vec{F}$ has zero curl everywhere then the flux of $\vec{F}$ through any closed surface S is zero.

12) T F The equation div(grad(f)) = 0 is an example of a partial differential equation for an unknown function $f(x,y,z)$.

13) T F The vector $(\vec{i}+\vec{j}) \times (\vec{i}-\vec{j})$ is the zero vector if $\vec{i} = \langle 1,0,0 \rangle$ and $\vec{j} = \langle 0,1,0 \rangle$.

14) T F If f is maximal under the constraint $g = c$, then the angle between $\nabla f(x,y)$ and $\nabla g(x,y)$ is zero.

15) T F Let L be the line $x = y, z = 0$ in the plane $\Sigma : z = 0$ and let P be a point. Then $d(P,L) \geq d(P,\Sigma)$.

16) T F The chain rule assures that $\frac{d}{dt} f(\vec{r}\,'(t)) = \nabla f(\vec{r}\,'(t)) \cdot \vec{r}\,''(t)$.

17) T F If K is a plane in space and P is a point not on K, there is a unique point Q on K for which the distance $d(P,Q)$ is minimized.

18) T F If $\vec{B}(t)$ is the bi-normal vector to an ellipse $\vec{r}(t)$ contained in the plane $x + y + z = 1$, then $\vec{B}$ is parallel to $\langle 1,1,1 \rangle$.

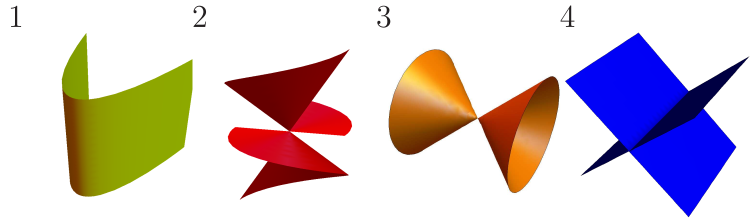
19) T F The parametrized surface $\vec{r}(u,v) = \langle u, v, u^2 + v^2 \rangle$ is everywhere perpendicular to the vector field $\vec{F}(x,y,z) = \langle x, y, x^2 + y^2 \rangle$.

20) T F Assume $\vec{r}(t)$ is a flow line of a vector field $\vec{F} = \nabla f$. Then $\vec{r}\,'(t) = \vec{0}$ if $\vec{r}(t)$ is located at a critical point of f.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

Surface	1-4
$x^2 - z^2 = 0$	
$\vec{r}(u,v) = \langle u\cos(v), u\sin(v), uv \rangle$	
$\vec{r}(u,v) = \langle u, u^2, v \rangle$	
$x^2 - y^2 = z^2$	



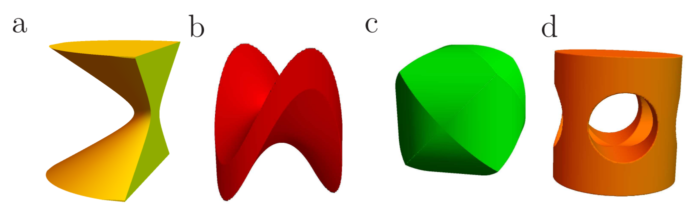
b) (2 points) Match the expressions. There is an exact match.

Integral	Enter A-D
$\int_a^b \int_c^d \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dudv$	
$\int_a^b \lvert\vec{r}\,'(t)\rvert\ dt$	
$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$	
$\int_a^b \int_c^d \lvert\vec{r}_u \times \vec{r}_v\rvert\ dudv$	

	Type of integral
A	line integral
B	flux integral
C	arc length
D	surface area

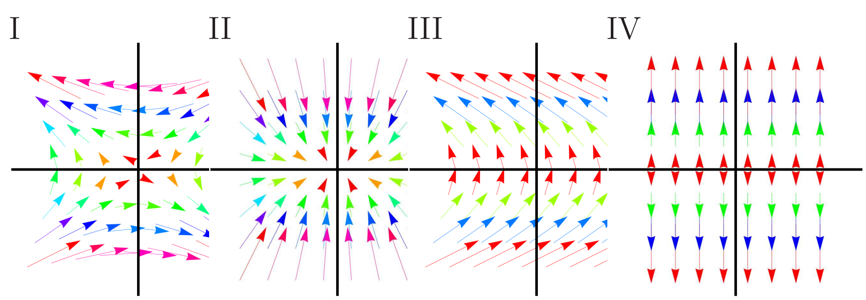
c) (2 points) Match the solids. There is an exact match.

Solid	a-d
$0 \leq x^2 - y^2 - z \leq 1$	
$x^2 + y^2 - z^2 \leq 1, x \geq 0$	
$x^2 + y^2 \leq 3, y^2 + z^2 \geq 1, x^2 + z^2 \geq 1$	
$x^2 + y^2 \leq 1, y^2 + z^2 \leq 1, x^2 + z^2 \leq 1$	



d) (2 points) The figures display vector fields in the plane. There is an exact match.

Field	I-IV
$\vec{F}(x,y) = \langle 0, 2y \rangle$	
$\vec{F}(x,y) = \langle -x, -2y \rangle$	
$\vec{F}(x,y) = \langle -2y, -x \rangle$	
$\vec{F}(x,y) = \langle -2y, 1 \rangle$	



e) (2 points) Match the partial differential equations with formulas and functions $u(t,x)$. There is an exact match.

Equation	1-3	A-C
wave		
heat		
Laplace		

	Formulas
1	$u_{tt} = -u_{xx}$
2	$u_t = u_{xx}$
3	$u_{tt} = u_{xx}$

	Functions
A	$u(t,x) = t + t^2 - x^2$
B	$u(t,x) = t + t^2 + x^2$
C	$u(t,x) = x^2 + 2t$

Problem 3) (10 points)

a) (2 points) Mark the statement or statements which have a zero answer.

Statement	Must be zero
The curl of the gradient $\nabla f(x,y,z)$ at $(1,1,1)$	
The divergence of the curl $\nabla \times \vec{F}(x,y,z)$ at $(1,1,1)$	
The flux of a gradient field $\nabla f(x,y,z)$ through a sphere	
The dot product of $\vec{F}(1,1,1)$ with the curl$(F)(1,1,1)$	
The divergence of a gradient field $\nabla f(x,y,z)$ at $(1,1,1)$	

b) (2 points) Two of the following statements do not make sense. Recall that "incompressible" means zero divergence everywhere and that "irrotational" means zero curl everywhere.

Statement	Makes no sense
The discriminant of the gradient field	
A conservative and incompressible vector field	
The flux of the gradient of a function through a surface	
The gradient of the curl of a vector field	

c) (2 points) Match the following formulas with the geometric object they describe. Fill in the blanks as needed.

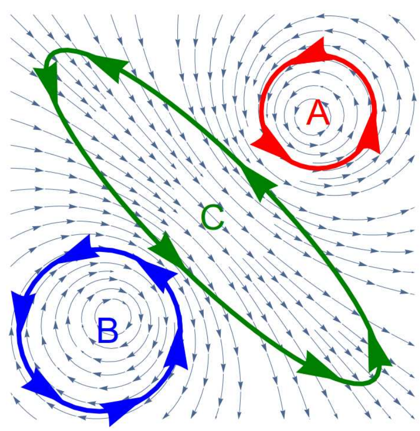
	Geometric object
A	unit tangent vector to a curve
B	unit normal vector to a surface
C	unit normal vector to a level curve
D	curvature of the curve
E	unit tangent vector to the surface

A-E	Formula
	$(\vec{r}_u \times \vec{r}_v)/\lvert\vec{r}_u \times \vec{r}_v\rvert$
	$\nabla f(x,y)/\lvert\nabla f(x,y)\rvert$
	$\vec{r}\,'(t)/\lvert\vec{r}\,'(t)\rvert$
	$(\vec{r}_u + \vec{r}_v)/\lvert\vec{r}_u + \vec{r}_v\rvert$
	$\lvert\vec{T}\,'(t)\rvert/\lvert\vec{r}\,'(t)\rvert$

d) (2 points) All three curves in the figure are oriented counter clockwise. Check whether in each of the three cases, the line integral is positive, negative or zero.

Check with a mark which applies. The line integral $\int_\gamma \vec{F} \cdot d\vec{r}$ is

γ	Positive	Negative	Zero
A			
B			
C			

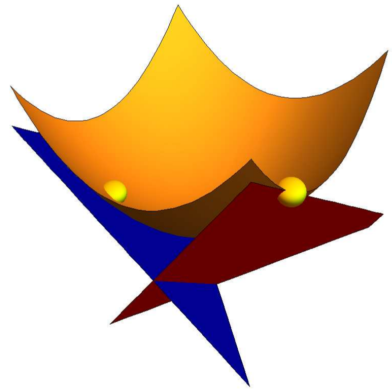


e) (2 points) Line or a plane? That is here the question.

U or V ?	Object
	$\vec{r}(0) + t\vec{B}(\vec{r}(0)) + s\vec{N}(\vec{r}(0))$
	$\vec{r}(0,0) + t\,\vec{r}_u \times \vec{r}_v$

	Dichotomy
U	is a normal plane to a curve
V	is a normal line to a surface

Problem 4) (10 points)

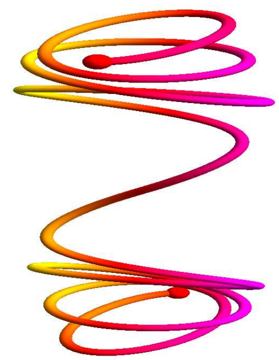


a) (6 points) Find the tangent planes at the points $A = (1,1,2)$ and $B = (-2,1,5)$ to the surface

$$x^2 + y^2 - z = 0 .$$

b) (4 points) Find the parametric equation of the line of intersection of these two tangent planes.

Problem 5) (10 points)



Compute the line integral of the vector field

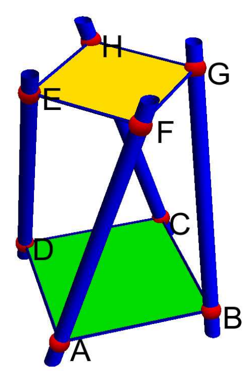
$$\vec{F}(x,y,z) = \langle 2x, 3y^3, 3z^2 \rangle$$

along the curve

$$\vec{r}(t) = \langle \sin(t)\cos(t^3), \cos(t)\sin(t^3), t \rangle$$

from $t = -\pi/2$ to $t = \pi/2$.

Problem 6) (10 points)



A monument has the form of a **twisted frustrum**. It is built from a base square platform $\{A = (-10,-10,0), B = (10,-10,0), C = (10,10,0), D = (-10,10,0)\}$ connected with an upper square $\{E = (-10,0,20), F = (0,-10,20), G = (10,0,20), H = (0,10,20)\}$ using cylinders of radius 1. Find the distance between the pillar going through AF and the pillar going through BG.

Problem 7) (10 points)

The monkey saddle gets a cameo: find the volume of the solid contained in the cylinder

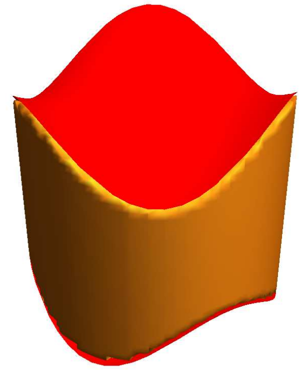
$$x^2 + y^2 < 1$$

above the surface

$$z = x^3 - 3xy^2 - 3$$

and below the surface

$$z = y^3 - 3yx^2 + 3 .$$

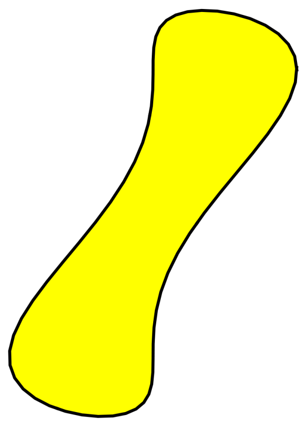


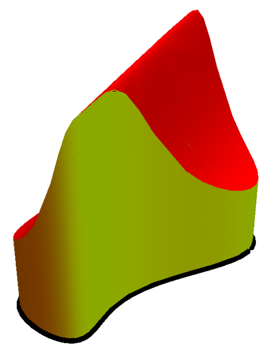
Problem 8) (10 points)

The volume of the solid above the region bound by the planar curve C : $\langle \cos^3(t), \sin(t) + \cos(t) \rangle$ and the graph of $f(x,y) = x^{-2/3}$ is given by the double integral

$$\int\int_G \frac{1}{\sqrt[3]{x^2}}\, dxdy .$$

Use the vector field $\vec{F} = \langle 0, 3x^{1/3} \rangle$ to compute this integral.





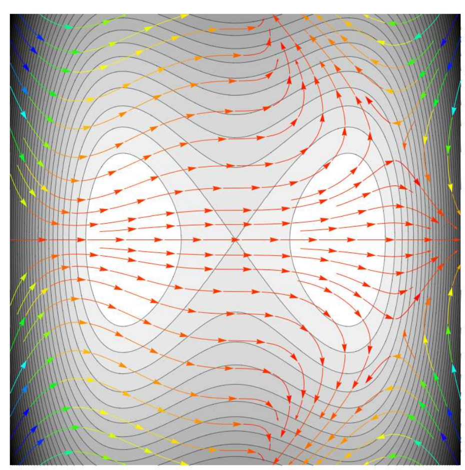
Problem 9) (10 points)

a) (8 points) The divergence of the vector field

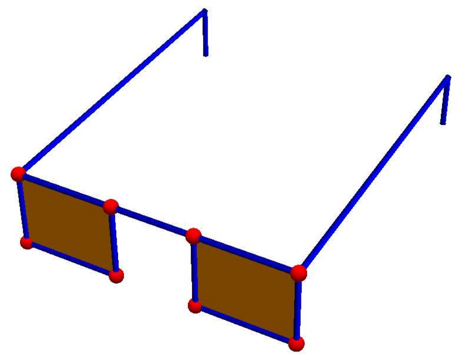
$$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle 1 - xy^2, 2 + 2yx^2 - yx^4 \rangle$$

is the function $f(x,y) = P_x + Q_y$. Find and classify the critical points of $f(x,y)$.

b) (2 points) We have seen in general that the gradient field $\nabla f(x,y)$ is perpendicular to level curves $\{f(x,y) = c\}$ and that $\nabla f(x,y)$ is the zero vector at maxima or minima. Is the vector field $\vec{F}(x,y)$ zero at a maximum of $f(x,y) = \text{div}(F)(x,y)$?



Problem 10) (10 points)

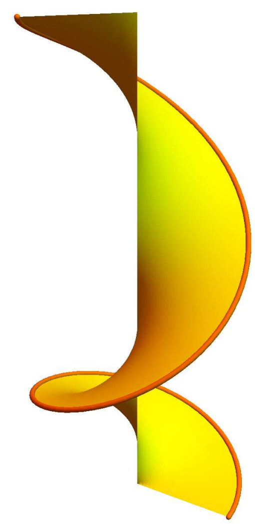


Economists know a **constraint duality principle**: "maximizing a first quantity while fixing the second is equivalent to minimizing the second when fixing the first". Let's experiment with that:

a) (5 points) Use the Lagrange method to find the **reading glasses** with maximal glass area $f(x,y) = 2xy$ and fixed frame material $4x + 15y = 120$.

b) (5 points) Use again the Lagrange method to find the reading glasses with minimal frame material $f(x,y) = 4x + 15y$ and fixed glass area $g(x,y) = 2xy = 120$.

Problem 11) (10 points)



a) (4 points) Find the arc length of the **helical curve**

$$\vec{r}(t) = \langle \cos(t), \sin(t), \frac{2t^{3/2}}{3} \rangle ,$$

where t goes from 0 to 9.

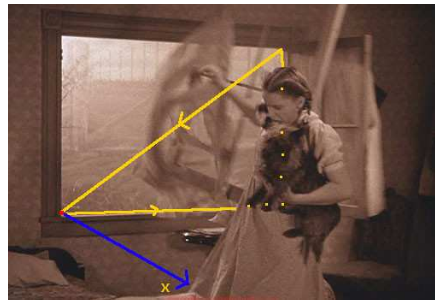
b) (3 points) Determine the angle between the velocity $\vec{r}\,'(t)$ and acceleration $\vec{r}\,''(t)$ at $t = 0$.

c) (3 points) Write down the surface area integral for the surface $\vec{r}(s,t) = \langle s\cos(t), s\sin(t), 2t^{3/2}/3 \rangle$ contained inside the cylinder $x^2 + y^2 \leq 1$ and between $0 \leq z \leq 18$ containing the previous curve in its boundary. You do not have to compute the integral but write down an expression of the form $\int_{\square}^{\square}\int_{\square}^{\square} f(s,t)\, dsdt$ with a function $f(s,t)$ you determine.

Problem 12) (10 points)

The tornado in the **wizard of Oz** induces the **force field**

$$\vec{F} = \langle \cos(x), -2x, y^3 + \sin(z^5) \rangle .$$

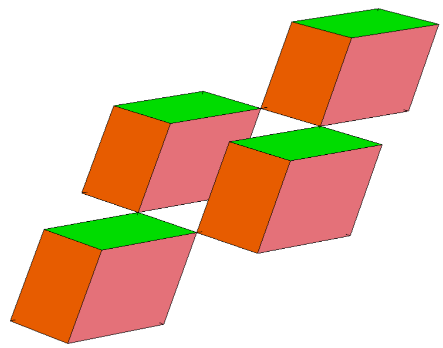


Dorothy's cardboard is picked up by the storm and pushed along the boundary of the triangle parametrized by

$$\vec{r}(u,v) = \langle 0, u, v \rangle$$

with $0 \leq u \leq 2$ and $0 \leq v \leq u/2$. Let C be the boundary of the triangle, oriented counter clockwise when looking from $(1,0,0)$ onto the window. Find the work $\int_C \vec{F} \cdot d\vec{r}$ which the tornado does onto the card board.

Problem 13) (10 points)



A solid E is the union of 4 congruent, non-intersecting parallelepipeds. One of them is spanned by the three vectors

$$\vec{u} = \langle 1,0,0 \rangle, \vec{v} = \langle 1,1,0 \rangle, \vec{w} = \langle 0,1,1 \rangle .$$

Find the flux of the vector field

$$\vec{F} = \langle 4x + y^{2014}, z^{2014}, x^{2014} \rangle + \text{curl}(\langle -y^{2014}, x^{2014}, 3^{2014} \rangle)$$

through the outwards oriented boundary surface of E.