THE EARLY HISTORY OF PARTIAL DIFFERENTIAL EQUATIONS AND OF PARTIAL DIFFERENTIATION AND INTEGRATION

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The general events associated with the evolution of the fundamental concepts of fluxions and the calculus are so very absorbing, that the history of the very specialized topic of partial differential equations and of partial differentiation and integration has not received adequate attention for the early period preceding Leonhard Euler's momentous contributions to this subject. The pre-Eulerian history of the partial processes of the calculus is difficult to trace, for the reason that there existed at that time no recognized symbolism nor technical phraseology which would distinguish the partial processes from the ordinary ones. In consequence, historians have disagreed as to the interpretation of certain passages in early writers. As we shall see, meanings have been read into passages which the writers themselves perhaps never entertained. In connection with fluxions certain erroneous *a priori* conceptions of their theory were entertained by some historians which would have been corrected, had these historians taken the precaution of proceeding more empirically and checking their pre-conceived ideas by reference to the actual facts.

Partial Processes in the writings of Leibniz and his immediate followers

Partial differentiation and partial integration occur even in ordinary processes of the calculus where partial differential equations do not occur. The simplest example of partial differentiation is seen in differentiating the product xy, where one variable is for the moment assumed to be constant, then the other. Leibniz used partial processes, but did not explicitly employ partial differential equations. He actually used special symbols, in a letter[1] to de l'Hospital in 1694, when he wrote "δm" for the partial derivative $\partial m/\partial x$, and "ϑm" for $\partial m/\partial y$; De l'Hospital used "δm" in his reply of March 2, 1695. As stated in his letter, Leibniz considers the integration of $bdx+cdy$, where b and c involve x and y, and seeks an equation $m=0$ where m also involves x and y. Differentiating $m=0$ yields him $\delta mdx+\vartheta mdy=0$. We have here a total differential equation. It follows, he says, that $b:c=\delta m:\vartheta m$ or $b\vartheta m=c\delta m$. In the analysis which follows this statement, Leibniz says that this last equation is to be satisfied *identically*. It is clear that in deriving the above total differential equation Leibniz differentiates partially, taking first x as an independent variable, then y as an independent variable. That the *identity* which follows was recognized by him as a partial differential equation is not clear. Such recognition would demand in case of an identity an abstract view point hardly

[1] Leibnizens Mathematische Schriften, vol. II, Berlin (1850), pp. 261, 270.

attributable to writers in the prelude period of the history of partial differential equations.

A recent writer claims that Leibniz did use partial differential equations: "Auf partielle Differentialgleichungen kommt Leibniz durch ein geometrisches Problem."[1] Reference is made to Leibniz's article[2] of 1694 in which he finds the envelope of the circles $x^2+y^2+b^2=2bx+ab$. Differentiating with reference to b as a variable parameter, Leibniz obtains $2bdb=2xdb+adb$. Eliminating b between $2b=2x+a$ and the given equation, he obtains as the required envelope the parabola $ax+(a^2/4)=y^2$. After studying Leibniz's introductory remarks, we feel that, in differentiating the two sides of the equation, he consciously kept both x and y constant and took b as an independent variable. We feel this notwithstanding the fact that he does not state this relation explicitly when differentiating. Leibniz did not call the equation $2b=2x+a$ by any special name. It is not a differential equation, but the process of partial differentiation is involved in its derivation.

Nor can we accept the validity of the recent claim[3] that Jakob Hermann used partial differentiation and partial differential equations in 1717, in special solutions of the celebrated problem of orthogonal trajectories to plane curves.[4] That problem, as ordinarily treated, does not give rise to partial differential equations, nor even to partial differentiation, except perhaps in the differentiation of implicit functions. The process, as followed by Hermann, consists in finding the total derivative dy/dx, introducing $-dx/dy$ in its place, and eliminating a parameter. Hermann solves four special cases, but does not give the equations to be differentiated in the form requiring partial differentiation.

However, Hermann did use partial differentiation on another occasion. Leibniz,[5] in a letter to John Bernoulli, describes a procedure which Hermann is reported to have explained to Chr. Wolf, and which clearly involves this process.

Partial differential equations stand out clearly in six examples on trajectories published in 1719 by Nicolaus Bernoulli (1695–1726),[6] the twenty-four year old son of John. He takes the curve $y^m=a^{m-1}x$, "cujus differentialis com-

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[1] Edmund Hoppe, Archiv für Geschichte der Mathematik, der Naturwissenschaften und der Technik, vol. 10 (1927), pp.161, 162. Hoppe refers to articles and letters of Leibniz in Acta eruditorum (1694), p. 311; Leibnizens Mathematische Schriften, vol. II, p. 166, vol. III (1855), pp. 967, 969. But we have not been able to find that Leibniz gives or solves partial differential equations in these articles.

[2] Acta eruditorum, 1694, p. 311.

[3] E. Hoppe, *loc. cit.*, p. 163.

[4] J. Hermann, Acta eruditorum (1717), p. 349.

[5] Leibniz's letter to John Bernoulli, July 26, 1716, Leibnizens Mathematische Schriften, vol. III, pp. 967, 968.

[6] Acta eruditorum (1719), p. 298. Republished in Johannis Bernoullo Opera omnia, Lausannae et Genevae, vol. II (1742), p. 399.

pleta est $my^{m-1}dy=(m-1)a^{m-2}x\ da+a^{m-1}\ dx$; hic $p=my^{m-1}:a^{m-1}=mx:y$, et $q=(1-m)x:a, \ldots$ ". Here we have the "complete" differentiation, followed by the two partial differential equations, in which $p=\partial x/\partial y$ and $q=\partial x/\partial a$.

Argument over a differential equation of Newton involving three variables

In his *Method of Fluxions* Newton solves differential equations and gives one example, $2\dot{x}-\dot{z}+\dot{y}x=0$, of a differential equation involving three variables. The fluxion $\dot{x}$ signifies our time-derivative dx/dt. On the European continent, some writers have interpreted this equation as a partial differential equation, other writers, as a total differential equation. We begin by quoting the passage in Newton:[1]

"The Resolution of the Problem will soon be dispatch'd, when the Equation involves three or more Fluxions of Quantities. For between any two of those Quantities any Relation may be assumed, when it is not determined by the State of the Question, and the Relation of their Fluxions may be found from thence; so that either of them, together with its Fluxion, may be extermin-ated. . . . . Let the Equation proposed be $2\dot{x}-\dot{z}+\dot{y}x=0$; that I may obtain the Relation of the Quantities x, y, and z, whose Fluxions $\dot{x}$, $\dot{y}$, and $\dot{z}$ are contained in the Equation; I form a Relation at pleasure between any two of them, as x and y, supposing that $x=y$ or $2y=a+z$, or $x=yy$ etc. But suppose at present $x=yy$ and therefore $\dot{x}=2\dot{y}y$. Therefore writing $2\dot{y}y$ for $\dot{x}$, and yy for x, the Equation proposed will be transform'd into this: $4\dot{y}y-\dot{z}+\dot{y}y^2=0$. And thence the relation between y and z will arise, $2yy+\frac{1}{3}y^3=z$. In which if x be written for yy and $x^{\frac{3}{2}}$ for y^3, we shall have $2x+\frac{1}{3}x^{\frac{3}{2}}=z$. So that among the infinite ways in which x, y, and z may be related to each other, one of them is here found, which is represented by these Equations, $x=yy$, $2y^2+\frac{1}{3}y^3=z$, and $2x+\frac{1}{3}x^{\frac{3}{2}}=z$."

The well known French writer on the calculus, Lacroix,[2] interpreted Newton's equation in three variables as a total differential equation.

On the other hand Weissenborn[3] says that Newton's problem was "nothing less than that of partial differential equations," in the treatment of which he was "not successful" since his solution is incorrect, as "one may see easily by trial." Weissenborn assigns no special reason for interpreting it as a partial differential equation any more than had Lacroix for calling it a total. That Newton's equation was "partial" was held also by the Swiss historian Heinrich Suter,[4] and the noted German historian Moritz Cantor[5] in the third volume of his

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[1] Sir Isaac Newton, *The Method of Fluxions*, translated into English by John Colson, London (1736), p. 41.

[2] S. F. Lacroix, *Traité du calcul différentiel et du calcul intégral*, 2d. ed., vol. II (1814), p. 691.

[3] Hermann Weissenborn, *Die Principien der höheren Analysis*, Halle (1856), p. 39.

[4] H. Suter, *Geschichte der Mathematischen Wissenschaften*, Zürich, vol. II (1875), p. 74.

[5] Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. III, 1898, p. 166.

Vorlesungen, but in the Preface Cantor retracts the statement, only to adhere to his original view in the second edition of that volume which appeared three years later.[1]

In direct opposition to Weissenborn's interpretation are E. Tischer[2] of Leipzig, Zeuthen[3] of Copenhagen, Eneström[4] of Stockholm and, very recently, Hoppe of Göttingen. Our own conclusion is in agreement with the last four writers; Newton's equation in three variables is a total differential equation. Newton treats this equation precisely as he does differential equations involving two variables, except that he assumes now a second relation $x=yy$ to exist, so that he can eliminate x and $\dot{x}$, and thereupon proceed as in case of two variables. Newton does not refer to any new principle involved in his equation in three variables. Moreover, Newton's solution of the equation is correct on the assumption that the differential equation is total, but incorrect on the assumption that it is partial. The conclusion is firmly established that neither Newton nor Leibniz in their published writings ever wrote down a partial differential equation and proceeded to solve it.

Erroneous conceptions about the theory of fluxions

While we agree with Tischer, Zeuthen and Hoppe that Newton's three-variable equation is a total differential equation, we do not agree at all with the reason which they assign for their conclusion. They base it on a preconceived erroneous conception according to which a partial differential equation is impossible on the Newtonian theory of fluxions, for the reason that Newton's fluxions are all time-derivatives and therefore exclude, as the critics state, any independent variable other than "time." Thus Tischer says on page 40 of his tract: "The concern here is not with a function of several independent variables, but with several functions of one and the same independent variable." Zeuthen states[5]: "Since the entire theory of fluxions rests upon the assumption of a single independent variable, he (Newton) remarks that in cases where an equation is given involving more than two variables with their fluxions, new relations between the variables may be introduced." We quote also Hoppe's statement: "When Newton has an equation in several fluents (we would say in variable magnitudes), that is, in x, y, z, u, etc., he derives the fluxions by assuming that

[1] *Op. cit.*, vol. 3, 2d. ed. (1901), p. 172.

[2] Ernst Tischer, *Ueber die Begründung der Infinitesimalrechnung durch Newton und Leibniz*, Jahresbericht des Nicolaigymnasiums in Leipzig, Leipzig (1896), pp. 37–39.

[3] H. G. Zeuthen, *Geschichte der Mathematik im XVI und XVII. Jahrhundert*, Deutsche Ausgabe, von R. Meyer, Leipzig (1903), p. 379; also Zeuthen in Bullet. de l'Acad. d. sc. de Danemark (1895), p. 263.

[4] G. Eneström, Bibliotheca mathematica, vol. IV (1903), p. 400; vol. 11 (1910–11), pp. 172–173.

[5] Zeuthen, *op. cit.*, p. 379. See also p. 358.

these variables are all functions of *one and the same variable magnitude*, for example, of the time. Then he marks the fluxion of x by $\dot{x}$, of y by $\dot{y}$, etc., so that $\dot{x}$ means in our notation dx/dt, $\dot{y}=dy/dt$, $\dot{z}=dz/dt$. Accordingly, if Newton had chosen to write his fluent equation briefly $f(x, y, z, \ldots)=u$, he would have meant by this, that $x, y, z, \ldots, u$ are functions of t, but not .... that u is a function of the independent variables, $x, y, z, \ldots$. The $x, y, z, \ldots, u$ were themselves functions of the variable t the time or of the temperature or some similar variable, and f signified only that an equation existed between these functions. If Newton wished to represent dy/dx he had to write $dy/dx=dy/dt\div dx/dt=\dot{y}\div\dot{x}=a:1$, where a is a magnitude measured according to the unit of the fluxions. ... In this theory of fluxions no path was open to partial differential equations."

These writers are correct in stating that all fluxions are time-derivatives, but where in Newton and other writers on fluxions is it stated that all fluxions must result from *contemporaneous* fluents? Why is it not possible to consider the velocity (fluxion) of u when only x changes, or when only y or z changes? These continental writers assume all fluxions to be contemporaneous and do not go to the trouble to see what the practice of British writers really was in this respect. To show the error of this contention it is sufficient to quote from Newton and other writers on fluxions where partial processes freely enter. To present our case convincingly, we shall go into considerable detail and thereby hope to make a contribution to the history of partial processes in England during the time of Newton and the eighteenth century.

Partial differentiation and partial integration in Newton

In the following quotation[1] from the *Method of Fluxions*, Newton explains the differentiation of an implicit function in x and y:

"If the relation of the flowing Quantities x and y be $x^3-ax^2+axy-y^3=0$; first dispose the Terms according to x, and then according to y, and multiply them in the following manner.

Mult.	$x^3 - ax^2 + axy - y^3$	$-y^3 + axy - ax^2 + x^3$
by	$\frac{3\dot{x}}{x} \cdot \frac{2\dot{x}}{x} \cdot \frac{\dot{x}}{x} \cdot 0$	$\frac{3\dot{y}}{y} \cdot \frac{\dot{y}}{y} \cdot 0$
makes	$3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y$ *	$-3\dot{y}y^2 + a\dot{y}x$ *

The sum of the Products is $3\dot{x}x^2-2a\dot{x}x+a\dot{x}y-3\dot{y}y^2+a\dot{y}x=0$, which Equation gives the Relation between the Fluxions $\dot{x}$ and $\dot{y}$. For if you take x at pleasure, the Equation $x^3-ax^2+axy-y^3=0$ will give y. Which being determined, it will be $\dot{x}:\dot{y}::3y^2-ax:3x^2-2ax+ay$."

[1] Sir Isaac Newton, *Method of Fluxions*, edited by John Colson (1736), p. 21.

Here clearly Newton allows x to vary, while y remains constant, and *vice versa*. In modern symbols, if $z=x^3-ax^2+axy-y^3$, then $\partial z/\partial x=3x^2-2ax+ay$, $\partial z/\partial y=ax-3y^2$. Such a procedure is a violation of the theory of fluxions as understood by Tischer, Zeuthen and Hoppe.

In the process of partial integration, Newton's statement is equally clear. He considers the problem: A fluxional equation being given, to find a fluent equation. "As this Problem is the Converse of the foregoing, it must be solved by proceeding in a contrary manner."[1] He solves several examples. Thus Newton clearly and fully explained partial differentiation and partial integration, but nowhere does he give a partial fluxional or differential equation.

Newton worked on two problems for which he published only conclusions, namely the problem of the solid of least resistance,[2] and the problem of the path of a ray of light in a heterogeneous medium.[3] The modern general treatment of these problems involves partial differential equations. Whether or not Newton himself used such equations we cannot profitably discuss here.

Other British writers giving partial differentiation and partial integration

We have not been able to discover the explicit use of partial processes in Maclaurin, Taylor and Stirling. Apparently using partial fluxions consciously, John Turner,[4] a friend of Thomas Simpson, in 1748, maximized $v^4x^3y^2z$ when $v+x+y+z=b$. "Expunging z, $b-v-x-y=1/v^4x^3y^2$. In fluxions $-\dot{v}-\dot{x}-\dot{y}=-2\dot{y}/y^3x^3v^4-3\dot{x}/y^2x^4v^4-4\dot{v}/y^2x^3v^5$; whence $\dot{y}=2\dot{y}/y^3x^3v^4$, $\dot{x}=3\dot{x}/y^2x^4v^4$, $\dot{v}=4\dot{v}/y^2x^3v^5$.

And $1/v^4x^3y^2=y/2=x/3=v/4=b-v-x-y$." Thereupon each unknown is found in terms of b. In this process a total fluxional equation is found first; thereupon the partial fluxion obtained when y alone varies, on the left side of the fluent equation, is equaled to the partial fluxion with respect to y, on the right side of the equation. Similarly for the partial fluxions with respect to x, and v, respectively. Practically the same problem is solved in the same way by William Emerson.[5]

Using the fluxional notation, John Playfair performs partial differentiation and partial integration in finding solids of greatest attraction.[6] Partial processes occur, of course, in books employing partial fluxional equations.

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[1] *Method of Fluxions*, p. 25.

[2] Newton's *Principia*, Bk. II, Prop. 34, Scholium. See a modern treatment in A. R. Forsyth, *Calculus of Variations*, Cambridge (1927), p. 340; O. Bolza, Bibliotheca Mathematica, vol. 13 (1912–13), p. 146.

[3] Newton's *Opticks*, Book II, Part III, Prop. 8. See a modern treatment of the problem in R. A. Herman, *Treasise on Geometrical Optics*, Cambridge (1900), p. 308.

[4] The Mathematical Questions proposed in the Ladies' Diary, by Thomas Leybourn, vol. 1, London (1817), p. 395.

[5] W. Emerson, *The Doctrine of Fluxions*, 3d. Ed., London (1768), pp. 174, 175.

[6] Transactions of the Royal Society of Edinburgh, vol. 6 (1812), pp. 195, 196.

Partial Differential Equations in Books on Fluxions

British writers on the history of the calculus have never claimed for themselves any share, however modest, in the development of partial differential equations before the nineteenth century. John Leslie[1] speaks of the "capital extension about the middle of the last century by what is termed the Calculus of Partial Differences, which applies with singular felicity to the solution of the most arduous and recondite physical problems.. . . The first specimen of this sort of Integration was given by Euler in 1734, but D'Alembert expanded the process in his Discourse on the *General Cause of the Winds*, which appeared in 1749.. . . " Similar statements are made by David Brewster in the *Edinburgh Encyclopaedia*, article "Fluxions."

Nevertheless, I have found a few occurrences of partial fluxional equations. In 1737, Thomas Simpson[2] derived the maximum of the expression $(b^3-x^3)(x^2z-z^3)\cdot(xy-y^2)$. He began: "First considering y as a variable, we have $x\dot{y}-2y\dot{y}=0$, or $y=\frac{1}{2}x \therefore xy-yy=\dfrac{xx}{4}$. By making z variable, we have $x^2\dot{z}-3z^2\dot{z}=0$, or $z=\dfrac{x}{\sqrt{3}}$. . . ." We see that, in obtaining the first differential equation, the fluxion of the given expression is found when y is the independent variable, z and x being taken to be constant; in obtaining the second equation, z is the independent variable, y and x being taken constant. If we introduce the letter u to represent the expression to be maximized, then, in modern symbols, the above analysis includes the process of finding $\partial u/\partial y=A(x-2y)$, $\partial u/\partial z=B(x^2-3y^2)$, where $A\equiv(b^3-x^3)(x^2y-z^3)$, and $B\equiv(b^3-x^3)(xy-y^2)$. There are thus obtained two simultaneous partial differential equations with two independent variables y and z, in which, for a maximum value of u, $\partial u/\partial y=0$ and $\partial u/\partial z=0$.

A similar problem, to find the minimum of an expression $xx+yy+zz$, when $ax+by+cz=d$, is answered in the *Ladies' Diary* for 1757–58 by Lionel Charlton of Whitley, and seems to indicate that partial processes were understood by the rank and file of British mathematicians. "Now seeing that any two of the quantities x, y, z may be varied independently of the other, we shall (by making x and y to flow, while z remains constant) have $a\dot{x}+b\dot{y}=0$ and $2x\dot{x}+2y\dot{y}=0$ . . . ." "In the same manner" he lets x and z flow, while y remains constant. This amounts to taking y and z as independent variables and x as a dependent variable. Observing that $\dot{x}/\dot{y}=\partial x/\partial y$, there are derived in the solution of this problem the simultaneous partial differential equations in two independent variables $x\partial x/\partial y+y=0$, $x\partial x/\partial z+z=0$, $a\partial x/\partial y+b=0$ and $a\partial x/\partial z+c=0$.

[1] John Leslie, "On the Progress of Mathematical and Physical Science, chiefly during the eighteenth Century" in the eighth edition of the Encyclopaedia Britannica, p. 715.

[2] Thomas Simpson, *Treatise of Fluxions*, London (1737), p. 35.

A later writer, Vince,[1] solves the problem, given $x+y+z=a$ and xy^2z^3 a maximum, to find x, y, z. Vince says: "Let us suppose such a value of y to remain constant, whilst x and z vary till they answer the conditions, and then $\dot{x}+\dot{z}=0$ and $z^3\dot{x}+3xz^2\dot{z}=0$. . . . Now let us suppose the value of z to remain constant, and x and y to vary . . . ."

Partial differential equations are rarities in English articles of the eighteenth century and in English books (with the exception of Waring's). Rigorous conditions for maxima and minima in expressions of more than two variables were not attempted.

Edward Waring

He was the only eighteenth century Englishman who wrote on partial differential equations other than the simplest types of the first order. He was a Senior wrangler and was described as "one of the strongest compounds of vanity and modesty." According to David Brewster, "His writings are the only mathematical works published in this country, until late years, that have kept pace with the improvements made in this science on the continent." Waring admitted that he "never could hear of any reader in England, out of Cambridge, who took the pains to read and understand" his writings. In his *Meditationes analyticae* he devoted to partial differential equations twenty-four pages (pp. 231–254) in the first edition (1776), and seventeen pages in the second edition (1785). A persistent student may master this topic as presented in the first edition, but in the second edition hope of conquest vanishes. There are here fewer examples; the statements of processes have a brevity and generality never attempted by earlier writers nor probably by later ones. On this subject Waring displayed less originality than in the theory of equations, algebraic curves, and the theory of numbers. In the introduction to the first edition of his *Meditationes analyticae* he expresses indebtedness on this subject to Clairaut, Euler, D'Alembert and Condorcet; in the second edition, also to Fontaine, Lagrange and Laplace, but nowhere does he give specific bibliographical references. We find that nearly all partial differential equations given in the first edition are found in Euler's *Institvtiones calcvli integralis*, Petropoli, 1770. Waring's presentation is of interest in showing how this subject is treated in the fluxional notation. We give in translation from his Latin the following definition (1st ed., p. 231): "Let the quantity $\left(\frac{\dot{V}}{\dot{x}}\right)$ enclosed in parentheses denote the value of the fluxion of V, where x alone is variable, divided by $\dot{x}$; that is, if $\dot{V}=p\dot{x}+q\dot{y}$, then $\left(\frac{\dot{V}}{\dot{x}}\right)$ denotes the quantity p, and $\left(\frac{\dot{V}}{\dot{y}}\right)$ denotes the quantity q." "By $\left(\frac{\ddot{V}}{\dot{y}\dot{x}}\right)$ is denoted the second fluxion of V, divided by $\dot{x}\dot{y}$, in the

[1] S. Vince, Principles of Fluxions, 1st American Edition, Philadelphia (1812), p. 19.

derivation of which at first x alone varies, then y alone varies." In our notation these derivations are $\partial V/\partial x$ and $\partial^2 V/\partial x \partial y$. The use of parentheses to distinguish partial derivatives from total derivatives was introduced by Euler. In the first edition, Waring treats briefly D'Alembert's equation arising in the famous problem of vibrating strings, $\left(\frac{\ddot{z}}{\dot{y}^2}\right) = a\left(\frac{\ddot{z}}{\dot{x}^2}\right)$, also thirteen other linear partial differential equations of the second order, two equations of the first order and one of the third. In the second edition the treatment is more cavalierly presented; the solutions of only eight equations of the second order are sketched, and three theorems are given relating to equations of the nth order, one for homogeneous equations with constant coefficients and x and y as independent variables, another for homogeneous equations with constant coefficients and any number of independent variables, and the third for linear partial differential equations. The following quotation (2nd ed., p. 299) exhibits the notation and Waring's style of presentation:

"Sit aequatio $P\left(\frac{\overset{n}{\dot{V}}}{\dot{x}^n}\right) + Q\left(\frac{\overset{n}{\dot{V}}}{\dot{x}^{n-1}\dot{y}}\right) + R\left(\frac{\overset{n}{\dot{V}}}{\dot{x}^{n-2}\dot{y}^2}\right) + \text{etc.} + P'\left(\frac{\overset{n-1}{\dot{V}}}{\dot{x}^{n-1}}\right) + Q'\left(\frac{\overset{n-1}{\dot{V}}}{\dot{x}^{n-2}\dot{y}}\right)$ $+ R'\left(\frac{\overset{n-1}{\dot{V}}}{\dot{x}\dot{y}^{n-2}}\right) + \text{etc.} + P''\left(\frac{\overset{n-2}{\dot{V}}}{\dot{x}^{n-2}}\right) + \text{etc.} + LV = 0$; ubi in singulis terminis V vel ejus fluxio unam solummodo habet dimensionem; in hâc aequatione pro V & ejus fluxionibus scribantur $e^v \times u$ & ejus correspondentes fluxiones, ubi v & u sunt functiones quantitatum x & y: functiones v & u pendent e functionibus P, Q, R, etc., P', Q', etc., P'', etc. quo magis simplices sunt priores functiones, eo magis plerumque simplices erunt posteriores."

SOME TETRAHEDRAL COMPLEXES

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1. Consider the lines s in space such that the feet of the perpendiculars dropped upon them from a fixed point A lie in a given plane α.

2. What is the configuration formed by the lines s which pass through a given point M? The foot U of the perpendicular AU from A upon s lies in the plane α and also on the sphere (AM) having the segment AM for diameter. Therefore the lines s passing through M project from M the circle of intersection of (AM) with the plane α. Thus: *The lines s which pass through a given point in space form, in general, a cone of second degree.*

3. The lines s which lie in a given plane μ may be obtained as follows. Take any point U on the line of intersection $\alpha\mu$ of the planes α, μ and erect, in μ, the perpendicular s to AU. Now if A' is the projection of A upon μ, $A'U$ is, by a well known theorem of elementary geometry, perpendicular to the line s. Thus the line s is a side of a right angle in the plane μ, the other side of which