11/4/2014 SECOND HOURLY PRACTICE XII **Math 21a, Fall 2014**

Name:

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MWF 10 Yong-Suk Moon
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TTH 10 Peter Smillie
TTH 10 Jeff Kuan
TTH 10 Yi Xie
TTH 11:30 Jeff Kuan
TTH 11:30 Jameel Al-Aidroos

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (30 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) T F $f(x, y)$ and $g(x, y) = f(x^2, y^2)$ have the same critical points.

2) T F If a function $f(x, y) = ax + by$ has a critical point, then $f(x, y) = 0$ for all (x, y).

3) T F Given 2 arbitrary points in the plane, there is a function $f(x, y)$ which has these points as critical points and no other critical points.

4) T F If (x_0, y_0) is the maximum of $f(x, y)$ on the disc $x^2+y^2 \leq 1$ then $x_0^2+y_0^2 < 1$.

5) T F There are no functions $f(x, y)$ for which every point on the unit circle is a critical point.

6) T F An absolute maximum (x_0, y_0) of $f(x, y)$ is also an absolute maximum of $f(x, y)$ constrained to a curve $g(x, y) = c$ that goes through the point (x_0, y_0).

7) T F If $f(x, y)$ has two local maxima on the plane, then f must have a local minimum on the plane.

8) T F There exists a function $f(x, y)$ of two variables which has no critical points at all.

9) T F If $f_x(x, y) = f_y(x, y) = 0$ for all (x, y) then $f(x, y) = 0$ for all (x, y) .

10) T F $(0, 0)$ is a local maximum of the function $f(x, y) = x^2 - y^2 + x^4 + y^4$.

11) T F If $f(x, y)$ has a local maximum at the point $(0, 0)$ with discriminant $D > 0$ then $g(x, y) = f(x, y) - x^4 + y^3$ has a local maximum at the point $(0, 0)$ too.

12) T F Every critical point (x, y) of a function $f(x, y)$ for which the discriminant D is not zero is either a local maximum or a local minimum.

13) T F If $(0, 0)$ is a critical point of $f(x, y)$ and the discriminant D is zero but $f_{xx}(0, 0) < 0$ then $(0, 0)$ can not be a local minimum.

14) T F In the second derivative test, one can replace the condition $D > 0, f_{xx} > 0$ with $D > 0, f_{yy} > 0$ to check whether a point is a local minimum.

15) T F The function $f(x, y) = (x^4 - y^4)$ has neither a local maximum nor a local minimum at $(0, 0)$.

16) T F It is possible to find a function of two variables which has no maximum and no minimum.

17) T F The value of the function $f(x, y) = \sqrt{1 + 3x + 5y}$ at $(-0.002, 0.01)$ can by linear approximation be estimated as $1 - (3/2) \cdot 0.002 + (5/2) \cdot 0.01$.

18) T F The function $f(x, y) = e^y x^2 \sin(y^2)$ satisfies the partial differential equation $f_{xxyyyxyy} = 0$.

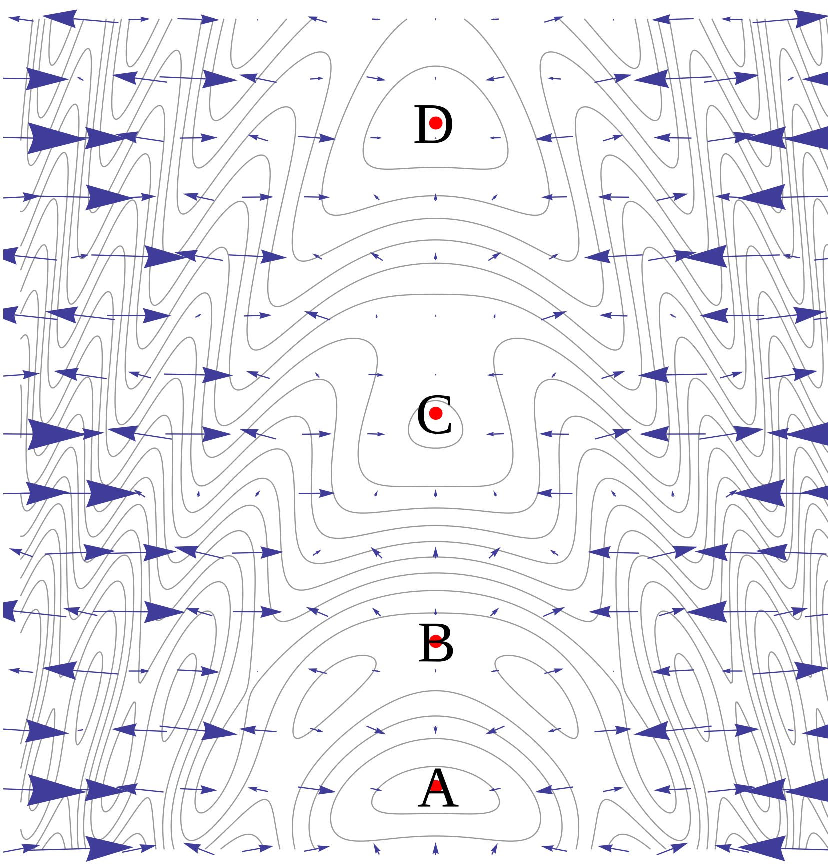
19) T F If $\vec{r}(t)$ is a curve with unit speed in the plane with $\vec{r}(0) = (0, 0)$ and $D_{\vec{r}\,'(0)} f(0, 0) = 0$, then $\frac{d}{dt} f(\vec{r}(t)) = 0$ at the time $t = 0$.

20) T F If a function $f(x, y)$ satisfies the partial differential equation $f_x^2 - f_y^2 = 0$, then f is the constant function.

Problem 2) (10 points)

a) (5 points) The picture below shows the contour map of a function $f(x, y)$ which has many critical points. Four of them are outlined for you on the y axes and are labeled A, B, C, D and ordered in increasing y value. The picture shows also the gradient vectors. Determine from each of the 4 points whether it is a local maximum, a local minimum or a saddle point. No justification is necessary in this problem.

Point	Max	Min	Saddle
D			
C			
B			
A			



b) (5 points)

Match the integrals with those obtained by changing the order of integration. No justifications are needed. Note that one of the Roman letters I)-V) will not be used, you have to chose four out of five.

Enter I,II,III,IV or V here.	Integral
	$\int_0^1 \int_{1-y}^1 f(x,y)\, dxdy$
	$\int_0^1 \int_y^1 f(x,y)\, dxdy$
	$\int_0^1 \int_0^{1-y} f(x,y)\, dxdy$
	$\int_0^1 \int_0^y f(x,y)\, dxdy$

I) $\int_0^1 \int_0^x f(x,y)\, dydx$
II) $\int_0^1 \int_0^{1-x} f(x,y)\, dydx$

III) $\int_0^1 \int_x^1 f(x,y)\, dydx$
IV) $\int_0^1 \int_0^{x-1} f(x,y)\, dydx$
V) $\int_0^1 \int_{1-x}^1 f(x,y)\, dydx$

Problem 3) (10 points)

When Ramanujan, the amazing India born mathematician was sick in the hospital in england and the English mathematician Hardy visited him, Ramanujan asked "whats up?" Hardy answered: "Nothing special. Even the number of the taxi cab was boring: 1729". Ramanujan answered: "No, that is a remarkable number. It is the smallest number, which can be written in two different ways as a sum of two perfect cubes. Indeed $1729 = 1^3 + 12^3 = 9^3 + 10^3$.

a) (5 points) Find the linearization $L(x,y,z)$ of the function $f(x,y,z) = x^3 + y^3 - z^3$ at the point $(9,10,12)$.

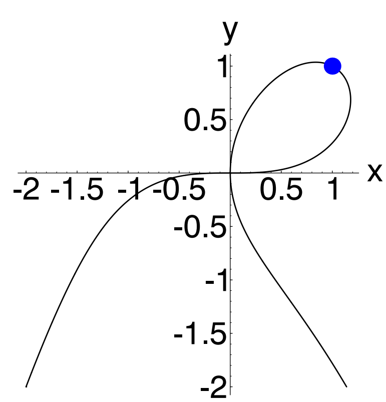
b) (5 points) Use the technique of linear approximation to estimate $9.001^3+10.02^3-12.001^3$. Since we are not all Ramanujans, you can leave the end result as a product and sum of numbers. For example, $234 \cdot 0.001 - 100 \cdot 0.002$ would be an acceptable end result.

Problem 4) (10 points)

Consider the equation

$$f(x,y) = 2y^3 + x^2y^2 - 4xy + x^4 = 0$$

It defines a curve, which you can see in the picture. Near the point $x = 1, y = 1$, the function can be written as a graph $y = y(x)$. Find the slope of that graph at the point $(1,1)$.



Problem 5) (10 points)

a) Find a point on the surface $g(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{8}{z} = 1$ which is locally closest to the origin.
b) Is this a global minimum? Hint: look at points $(x, y, z) = (1, -1/n, 8/n)$ where n is an integer.

Problem 6) (10 points)

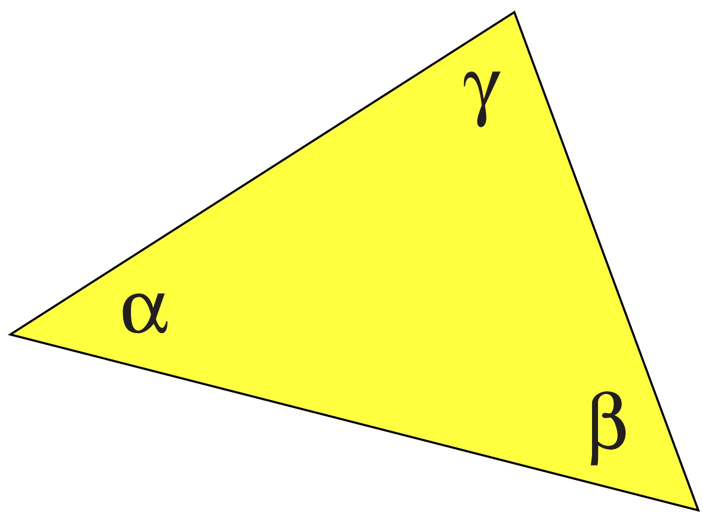
Find all extrema of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and characterize them. Do you find a absolute maximum or absolute minimum among them?

Problem 7) (10 points)

What is the shape of the triangle with angles α, β, γ for which

$$f(\alpha, \beta, \gamma) = \log(\sin(\alpha)\sin(\beta)\sin(\gamma))$$

is maximal?



Problem 8) (10 points)

Let $g(x, y)$ be the distance from a point (x, y) to the curve $x^2 + 2y^2 + y^4/10 = 1$. Show that g is a solution of the partial differential equation

$$f_x^2 + f_y^2 = 1$$

outside the curve.

Hint: no computations are needed. The shape of the curve is pretty much irrelevant. What does the PDE say about the gradient ∇f?

Remark: This problem only needs thought. Use it as a "pillow problem" that is think about it before going to sleep. By the way, the PDE is called **eiconal equation**. It describes wave fronts in optics.

Problem 9) (10 points)

a) (6 points) Find all critical points of $f(x,y) = 3xe^y - e^{3y} - x^3$ and classify them.

b) (4 points) Does the function have a absolute maximum or absolute minimum? Make sure to justify also this answer.

Problem 10) (10 points)

a) (5 points) Integrate $f(x,y) = x^2 - y^2$ over the unit disk $\{x^2 + y^2 \leq 1\}$.

b) (5 points) An evil integral!

$$\int_0^1 \int_0^{\sqrt{1-\theta^2}} r^2 \, dr d\theta \ .$$