PARTIAL DIFFERENTIAL EQUATIONS

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FUNCTIONS OF TWO VARIABLES. We consider functions \( f(x,t) \) in two variables. Thinking about the variable \( t \) as time, we can think of the functions of one variables \( x \rightarrow f(x,t) \) as they evolve in time. The describing equation will now be a partial differential equation (PDE), a differential equation which involves the derivatives with respect to both space \( x \) and time \( t \). The function \( f(x,t) \) could denote the temperature of a stick or the height of a water wave at position \( x \) and time \( t \).

PARTIAL DERIVATIVES. We write \( f_x(x,t) \) and \( f_t(x,t) \) for the partial derivatives with respect to \( x \) or \( t \). The notation \( f_{xx}(x,t) \) means that we differentiate twice with respect to \( x \).

Example: for \( f(x,t) = \cos(x+4t^2) \), we have
- \( f_x(x,t) = -\sin(x+4t^2) \)
- \( f_t(x,t) = -8t\sin(x+4t^2) \)
- \( f_{xx}(x,t) = -\cos(x+4t^2) \)

One also uses the notation \( \frac{\partial f(x,t)}{\partial x} \) for the partial derivative with respect to \( x \). Tired of all the "partial derivative notation"? We usually write \( f_t(x,y) \) or \( f_x(x,y) \) in this handout. This is an official abbreviation in the scientific literature.

PARTIAL DIFFERENTIAL EQUATIONS. A partial differential equation is an equation for an unknown function \( f(x,t) \) in which at least two different partial derivatives occur.

- \( f_t(x,t) + f_{xx}(x,t) = 0 \) with \( f(x,0) = \sin(x) \) has a solution \( f(x,t) = \sin(x-t) \).
- \( f_t(x,t) = f(x,t) \) has the solution \( f(x,0)e^t \). The equation is not a PDE. Why not?
- \( f_{xx}(x,t) = f_t(x,t) = 0 \) has a solution \( f(x,t) = \sin(x-t) + \sin(x+t) \). Check it!

EXAMPLE: THE HEAT EQUATION. The temperature distribution \( f(x,t) \) in a metal wire satisfies the heat equation

\[
f_t(x,t) = \mu f_{xx}(x,t)
\]

This PDE tells that the rate of change of the temperature at the point \( x \) is proportional to the second space derivative of \( f(x,t) \) at \( x \). A function \( f(x) = f(0,0) \) defines an initial temperature distribution. The constant \( \mu \) depends on the heat conductivity of the material. Metals for example conduct heat well and have a large \( \mu \).

VISUALIZATION. We can plot the graph of the function \( f(x,t) \) or plot the temperature distribution for different times \( t \).

EXAMPLE: THE WAVE EQUATION. The height of a wave \( f(x,t) \) at time \( t \) and position \( x \) satisfies the wave equation

\[
f_{tt}(x,t) = c^2 f_{xx}(x,t)
\]

where \( c \) is a constant, the speed of the waves.

VISUALIZATION. We can plot the wave height \( f(x,t) \) as a function of \( x \) for different but fixed times \( t \).

EXAMPLE: THE BURGERS EQUATION. If waves approach the shore, their dynamics changes. Low amplitude waves slow down and high altitude waves move faster. Additionally, waves start to dissipate and lose energy. A model is the Burgers equation

\[
f_t(x,t) + f(x,t)f_x(x,t) = \mu f_{xx}(x,t),
\]

This partial differential equation can have shocks: the waves break. You see that at the beach. With positive \( \mu \), one can give explicit traveling waves \( f(t,x) = (1 + e^{-t/4\nu})^{-1} \). Waves \( f(t,x) = \frac{\sqrt{\nu t}}{1 + \sqrt{\nu t}} e^{-t/4\nu} \) become discontinuous at \( t = 1 \).

VISUALIZATION. Again we can plot the wave functions \( f(x,t) \) for fixed times \( t \).

TO THE DERIVATION OF THE HEAT EQUATION. The temperature \( f(x,t) \) is proportional to the kinetic energy at \( x \). Divide the stick into \( n \) adjacent cells and assume that in each time step, a fraction of the particles moves randomly to the right or to the left. If \( f_k(t) \) is the energy of particles in cell \( k \) at time \( t \), then the energy of particles at time \( t + 1 \) is proportional to the sum of \( f_{k+1}(t) - f_k(t) \) and \( f_{k-1}(t) - f_k(t) \) which is \( (f_{k+1}(t) - 2f_k(t) + f_{k-1}(t)) \). This is a discrete version of the second derivative because \( \frac{d^2}{dx^2} f(x,t) \approx (f(x+dx,t) - 2f(x,t) + f(x-dx,t)) \).

TO THE DERIVATION OF THE WAVE EQUATION. A wave can be modeled by \( n \) particles linked by springs. Assume that the water particles move up and down only. If \( f_i(t) \) is the height of the particles, then the right particle pulls with a force \( f_{i+1} - f_i \), the left particle with a force \( f_{i-1} - f_i \). Again, \( (f_{i-1}(t) - 2f_i(t) + f_{i+1}(t)) \) is a discrete version of the second derivative \( f_{xx} \). By Newton’s law, the acceleration \( f_t(t,x) \) at position \( x \) is proportional to \( f_{xx} \).

TO THE DERIVATION OF BURGERS EQUATION. Assume \( \mu = 0 \) for a moment. If the wave \( f \) has height close to \( c \), we see that \( f_t(x,t) + cf_x(x,t) \) which has the solution \( f(x,t) = f(x-ct,0) \). The waves travel forward with a speed which depends on the height of the wave. Higher waves travel faster. The additional term \( \mu f_{xx} \) plays the same role as in the heat equation. The potential energy (which is proportional to the height of the wave) dissipates into the neighborhood.