Name:

- Please mark the box to the left which lists your section.
- Do not detach pages from this exam packet or un staple the packet.
- Show your work! Answers without reasoning cannot be given credit, except for the TF and multiple choice problems.
- Please write neatly. Answers which the grader cannot read will not receive credit.
- No notes, books, calculators, computers, or other electronic aids can be used.
- All unspecified functions mentioned in this exam are assumed to be smooth: you can differentiate as many times as you want with respect to any variables.
- You have 90 minutes to complete your work.

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Problem 1) TF questions (30 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) T F  
   \( f(x, y) \) and \( g(x, y) = f(x^2, y^2) \) have the same critical points.

Solution:
The function \( g \) has always \((0, 0)\) as a critical point, even if \( f \) has not.

2) T F
   If a function \( f(x, y) = ax + by \) has a critical point, then \( f(x, y) = 0 \) for all \((x, y)\).

Solution:
At a critical point the gradient is \((a, b) = (0, 0)\), which implies \( f = 0 \).

3) T F
   \( f_{yyxx} = f_{xyyx} \) for \( f(x, y) = \sin(\cos(y + x^{14}) + \cos(x)) \).

Solution:
Follows from Clairot’s theorem.

4) T F
   Given 2 arbitrary points in the plane, there is a function \( f(x, y) \) which has these points as critical points and no other critical points.

Solution:
Connect the two points with a line and take this height as the x-axes, centered at the midpoint and with units such that the two points have coordinates \((-1, 0), (1, 0)\). The function \( f(x, y) = -y^2(x^3 - 1) \) has the two points as critical points. One is a local max, the other is a saddle point.

5) T F
   It is possible that for some unit vector \( u \), the directional derivative \( D_u f(x, y) \)
   is zero even though the gradient \( \nabla f(x, y) \) is nonzero.

Solution:
Just go in to the direction tangent to the level curve.

6) T F
   If \((x_0, y_0)\) is the maximum of \( f(x, y) \) on the disc \( x^2 + y^2 \leq 1 \) then \( x_0^2 + y_0^2 < 1 \).
Solution:
The maximum could be on the boundary.

7) T F

The linear approximation \( L(x, y, z) \) of the function \( f(x, y, z) = 3x + 5y - 7z \) at \((0, 0, 0)\) satisfies \( L(x, y, z) = f(x, y, z) \).

Solution:
\( f(0, 0, 0) = 0 \) and \( \nabla f(0, 0, 0) = (3, 5, -7) \).

8) T F

If \( f(x, y) = \sin(x) + \sin(y) \), then \( -\sqrt{2} \leq D_u f(x, y) \leq \sqrt{2} \).

Solution:
\( |D_u f| \leq ||\nabla f|| \leq \sqrt{2} \).

9) T F

There are no functions \( f(x, y) \) for which every point on the unit circle is a critical point.

Solution:
There are many rotationally symmetric functions with this property.

10) T F

An absolute maximum \((x_0, y_0)\) of \( f(x, y) \) is also an absolute maximum of \( f(x, y) \) constrained to a curve \( g(x, y) = c \) that goes through the point \((x_0, y_0)\).

Solution:
The Lagrange multiplier vanishes in this case.

11) T F

If \( f(x, y) \) has two local maxima on the plane, then \( f \) must have a local minimum on the plane.

Solution:
Look at a camel type surface. It has a saddle between the local maxima.

12) T F

The acceleration of the curve \( \vec{r}(t) = (\cos(t), \sin(t), t) \) at time \( t = 0 \) is 1.
Solution:
The acceleration is a vector.

13)  True/False

There exists a function \( f(x, y) \) of two variables which has no critical points at all.

Solution:
True. Every non-constant linear function for example.

14)  True/False

If \( f_x(x, y) = f_y(x, y) = 0 \) for all \((x, y)\) then \( f(x, y) = 0 \) for all \((x, y)\).

Solution:
False, \( f \) could be constant.

15)  True/False

(0, 0) is a local maximum of the function
\[
 f(x, y) = x^2 - y^2 + x^4 + y^4.
\]

Solution:
(0, 0) is a saddle point.

16)  True/False

If \( f(x, y) \) has a local maximum at the point \((0, 0)\) with discriminant \( D > 0 \) then
\[
 g(x, y) = f(x, y) - x^4 + y^3
\]
has a local maximum at the point \((0, 0)\) too.

Solution:
Adding \( x^4 + y^3 \) does not change the first and second derivatives.

17)  True/False

The value of the function
\[
 f(x, y) = \sqrt{1 + 3x + 5y}
\]
at \((-0.002, 0.01)\) can by linear approximation be estimated as
\[
1 - (3/2) \cdot 0.002 + (5/2) \cdot 0.01.
\]

Solution:
Use formula for \( L(x, y) \).

18)  True/False

The gradient of \( f \) at a point \((x_0, y_0, z_0)\) is tangent to the level surface of \( f \) which contains \((x_0, y_0, z_0)\).
Solution:
It is a basic and important fact that $\nabla f$ is perpendicular to the level surface.

19) \( T \) \( F \) If $D_\vec{v}f(1,1) = 0$ for all vectors $\vec{v}$, then $(1,1)$ is a critical point of $f(x,y)$.

Solution:
Especially, $D_\nabla f(f) = |\nabla f|^2 = 0$ so that $\nabla f = (0,0,0)$.

20) \( T \) \( F \) The function $u(x,t) = x^3 + t^3$ satisfies the wave equation $u_{tt} = u_{xx}$.

Solution:
Just differentiate.

21) \( T \) \( F \) Every critical point $(x,y)$ of a function $f(x,y)$ for which the discriminant $D$ is not zero is either a local maximum or a local minimum.

Solution:
The second derivative test give for negative $D$ that we have a saddle point.

22) \( T \) \( F \) The function $f(x,y) = e^y x^2 \sin(y^2)$ satisfies the partial differential equation $f_{xxyy}y_{yy} = 0$.

Solution:
By Clairots theorem, we can have all three $x$ derivatives at the beginning.

23) \( T \) \( F \) If $(0,0)$ is a critical point of $f(x,y)$ and the discriminant $D$ is zero but $f_{xx}(0,0) < 0$ then $(0,0)$ can not be a local minimum.

Solution:
If $f_{xx}(0,0) < 0$ then on the $x$-axis the function $g(x) = f(x,0)$ has a local maximum. This means that there are points close to $(0,0)$ where the value of $f$ is larger.

24) \( T \) \( F \) In the second derivative test, one can replace the condition $D > 0, f_{xx} > 0$ with $D > 0, f_{yy} > 0$ to check whether a point is a local minimum.
Solution:
True. If $f_{xx}f_{yy} - f_{xy}^2 > 0$, then $f_{xx}$ and $f_{yy}$ must have the same signs.

25) $\boxed{T \quad F}$ The gradient $(2x, 2y)$ is perpendicular to the surface $z = x^2 + y^2$.

Solution:
The surface is the graph of a function $f(x, y)$. While the gradient of $f$ is perpendicular to the level curve of $f$, it is only the projection of the gradient to the function $g(x, y, z) = f(x, y) - z$. The later is perpendicular to the surface.

26) $\boxed{T \quad F}$ If $f(x, t)$ satisfies the Laplace equation $f_{xx} + f_{tt} = 0$ and simultaneously the wave equation $f_{xx} = f_{tt}$, then $f(x, t) = ax + bt + c$.

Solution:
Take $f(x, t) = xt$. (Here is how we get the general solution: From the two equations, we get $f_{xx} = 0$ and $f_{tt} = 0$. From $f_{xx} = 0$, we obtain that $f(x, t) = a(t)x + c(t)$. From $f_{tt} = 0$, we obtain $a(t)$ and $c(t)$ are linear in $t$. Therefore the general solution is $f(x, t) = atx + bt + cx + c$).

27) $\boxed{T \quad F}$ The function $f(x, y) = (x^4 - y^4)$ has neither a local maximum nor a local minimum at $(0, 0)$.

Solution:
The function is both smaller and bigger than $f(0, 0)$ for points near $(0, 0)$.

28) $\boxed{T \quad F}$ It is possible to find a function of two variables which has no maximum and no minimum.

Solution:
There are many linear functions like that.

29) $\boxed{T \quad F}$ The value of the function $f(x, y) = e^xy$ at $(0.001, -0.001)$ can by linear approximation be estimated as $-0.001$. 

Solution:
Because the gradient at $(0,0)$ is $(0,1)$ and $f(0,0) = 0$, the linear approximation is $L(x,y) = y$.

30) \text{F}

For any function $f(x,y,z)$ and any unit vectors $u,v$, one has the identity $D_{u \times v}f(x,y,z) = D_u f(x,y,z)D_v f(x,y,z)$.

Solution:
The directional derivative in the $u \times v$ direction has nothing to do with the directional derivatives into the other directions. An example, $u = (1,0,0), v = (0,1,0), f(x,y,z) = x + y$ is an example, where $D_{u \times v} f(x,y,z) = 0$ but $D_u f = 1, D_v f = 1$.

Problem 2) (10 points)

Match the parametric surfaces with their parameterization. No justification is needed.
Problem 3) (10 points)

a) Show that for any differentiable function \( g(x) \), the function \( u(x, y) = g(x^2 + y^2) \) satisfies the partial differential equation \( yu_x = xu_y \).

b) Assuming \( g'(5) \neq 0 \), let \( u \) be the function defined in a). Find the unit vector \( \vec{v} \) in the direction of maximal increase at the point \( (x, y) = (2, 1) \).
Solution:
a) Just differentiate:
\[ yu_x = yg'(x^2 + y^2)2x = 2x yg'(x^2 + y^2) \]
\[ xu_y = xg'(x^2 + y^2)2y = 2y xg'(x^2 + y^2) \]
These two expressions are the same.
b) The direction of maximal increase points into the direction of the gradient of \( u \) which is \( \nabla u(x,y) = (g'(x^2 + y^2)2x, g'(x^2 + y^2)2y) \).
At the point \( (x,y) = (2,1) \) we have \( (g'(5)4, g'(5)2) \). If we normalize that, we obtain \( \vec{v} = (4, 2)/\sqrt{20} \).

Problem 4) (10 points)

Which point on the surface \( g(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{8}{z} = 1 \) is closest to the origin?

Solution:
This is a Lagrange problem. One wants to minimize \( f(x, y, z) = x^2 + y^2 + z^2 \) under the constraint \( g(x, y, z) = 1 \). The Lagrange equations are

\[
\begin{align*}
\frac{-1}{x^2} &= 2\lambda x \\
\frac{-1}{y^2} &= 2\lambda y \\
\frac{-8}{z^2} &= 2\lambda z \\
\frac{1}{x} + \frac{1}{y} + \frac{8}{z} &= 1
\end{align*}
\]

The first two equations show \( x = y \), the first and third equations show \( 8/z^3 = 1/x^3 \) or \( z = 2x \). Plugging this into the last equation gives \( 2/x + 8/(2x) = 1 \) or \( x = 6, y = 6, z = 12 \). \( (x, y, z) = (6, 6, 12) \).
There is an interesting twist to this problem (as noted by one of the students Jacob Aptekar): consider the points \( (x, y, z) = (1, -1/n, 8/n) \), where \( n \) is a large integer, One can check that these points lie on the surface \( g(x, y, z) = 1 \). Their distance to the origin however decreases to 1 if \( n \) goes to infinity. So the point \( (6, 6, 12) \), while a local minimum is not a global minimum.

Problem 5) (10 points)

Find all extrema of the function \( f(x, y) = x^3 + y^3 - 3x - 12y + 20 \) on the plane and
characterize them. Do you find a absolute maximum or absolute minimum among them?

**Solution:**
The critical points satisfy \( \nabla f(x, y) = (0, 0) \) or \( (3x^2 - 3, 3y^2 - 12) = (0, 0) \). There are 4 critical points \((x, y) = (\pm 1, \pm 2)\). The discriminant is \( D = f_{xx}f_{yy} - f_{xy}^2 = 36xy \) and \( f_{xx} = 6x \).

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<th>D</th>
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<th>classification</th>
<th>value</th>
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<tr>
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<td>-6</td>
<td>maximum</td>
<td>38</td>
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<tr>
<td>(-1, 2)</td>
<td>-72</td>
<td>-6</td>
<td>saddle</td>
<td>6</td>
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<tr>
<td>(1, -2)</td>
<td>-72</td>
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<td>saddle</td>
<td>34</td>
</tr>
<tr>
<td>(1, 2)</td>
<td>72</td>
<td>6</td>
<td>minimum</td>
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Note that there are no global (= absolute) maxima nor global minima because the function takes arbitrarily large and small values. For \( y = 0 \) the function is \( g(x) = f(x, 0) = x^3 - 3x + 20 \) which satisfies \( \lim_{x \to \pm \infty} g(x) = \pm \infty \).

Problem 6) (10 points)

Find all the critical points of \( f(x, y) = \frac{x^5}{5} - \frac{x^2}{2} + \frac{y^3}{3} - y \) and indicate whether they are local maxima, local minima or saddle points.

**Solution:**
\( \nabla f(x, y) = (x^4 - x, (y^2 - 1)) = (0, 0) \) so that the critical points are \((0, 1), (0, -1), (1, 1), (1, -1)\). We have \( D = (4x^3 - 1)2y \) and \( f_{xx} = 4x^3 - 1 \).

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<td>( D = -2 )</td>
<td>-</td>
<td>saddle</td>
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<tr>
<td>(0, -1)</td>
<td>( D = 2 )</td>
<td>-1</td>
<td>local max</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>( D = 6 )</td>
<td>3</td>
<td>local min</td>
</tr>
<tr>
<td>(1, -1)</td>
<td>( D = -6 )</td>
<td>-</td>
<td>saddle</td>
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Problem 7) (10 points)

Use the technique of linear approximation to estimate \( f(0.003, -0.0001, \pi/2 + 0.01) \) for 

\[
  f(x, y, z) = \cos(xy + z) + x + 2z.
\]
Solution:
\[ L(x, y) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) \]
\[ f(x_0, y_0, z_0) = \cos(\pi/2) + \pi = \pi \]
\[ a = f_x(x_0, y_0, z_0) = -0 \sin(\pi/2) + 1 = 1 \]
\[ b = f_y(x_0, y_0, z_0) = -0 \sin(\pi/2) = 0 \]
\[ c = f_z(x_0, y_0, z_0) = -\sin(\pi/2) + 2 = 1 \]
\[ L(x, y) = \pi + 0.003 \cdot 1 + -0.0001 \cdot 0 + 0.01 \cdot 1 = \pi + 0.013. \]

Problem 8) (10 points)

Find the equation \( ax + by + cz = d \) for the tangent plane to the level surface of
\[ f(x, y, z) = \cos(xy + z) + x + 2z \]
(same function as in last problem) which contains the point \((0, 0, \pi/2)\).

Solution:
We have \( \nabla f(0, 0, \pi/2) = (1, 0, 1) \) so that the plane is \( x + z = \pi/2 \)

Problem 9) (10 points)

What is the shape of the triangle with angles \( \alpha, \beta, \gamma \) for which
\[ f(\alpha, \beta, \gamma) = \log(\sin(\alpha) \sin(\beta) \sin(\gamma)) \]
is maximal?
Solution:
The Lagrange equations are $\cot(\alpha) = \lambda$, $\cot(\beta) = \lambda$, $\cot(\gamma) = \lambda$. Because $\alpha, \beta, \gamma$ are all in $[0, \pi]$, we conclude that all are the same. From the last equation follows $\alpha = \beta = \gamma = \pi/3$ and $\sin(\alpha) \sin(\beta) \sin(\gamma) = (\sqrt{3}/2)^3$. 