At a local maximum \((x_0, y_0)\) of \(f(x, y)\), one has \(f_{yy}(x_0, y_0) \geq 0\).

**False.** At a local maximum, \(f_{yy} \leq 0\).

If \(R\) is the region bounded by \(x^2 + 4y^2 = 1\) then \(\int_R xy^4 \, dx \, dy < 0\).

**False.** The integral is zero because the integral on \(R \cap \{ x > 0 \}\) is the negative of \(R \cap \{ x < 0 \}\).

The vector \(\langle 2x, 2y \rangle\) is perpendicular to the surface \(z = x^2 + y^2\).

**False.** The surface is the graph of a function \(f(x, y)\). While the gradient of \(f\) is perpendicular to the level curve of \(f\), it is only the projection of the gradient to the function \(g(x, y, z) = f(x, y) - z\). The latter is perpendicular to the surface.

The equation \(f(x, y) = k\) implicitly defines \(x\) as a function of \(y\) and \(\frac{dx}{dy} = \frac{\partial f}{\partial y} / \frac{\partial f}{\partial x}\).

Almost right, the sign is wrong.

\(f(x, y) = \sqrt{(16 - x^2 - y^2)}\) has both an absolute maximum and an absolute minimum on its domain of definition.

The domain of definition is the disc \(x^2 + y^2 \leq 16\). The maximum 4 is in the center the absolute minimum 0 at the boundary.

If \((x_0, y_0)\) is a critical point of \(f(x, y)\) under the constraint \(g(x, y) = 0\), and \(f_{xy}(x_0, y_0) < 0\), then \((x_0, y_0)\) is a saddle point.

The point \((x_0, y_0)\) does not need to be a critical point of \(f\) at all.

The vector \(r_u(u, v)\) of a parameterized surface \((u, v) \mapsto r(u, v) = (x(u, v), y(u, v), z(u, v))\) is normal to the surface.

The vector is always tangent to the surface.

The identity \(\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) \, dy \, dx = \int_0^{\pi/2} \int_0^1 r^2 \, d\theta \, dr\) holds.

The area element \(d\theta \, dr\) should be replaced by \(r \, d\theta \, dr\). So, the right hand side should be \(\int_0^{\pi/2} \int_0^1 r^3 \, d\theta \, dr\).

\(f(x, y)\) and \(g(x, y) = f(x^2, y^2)\) have the same critical points.

The function \(g\) has always \((0, 0)\) as a critical point, even if \(f\) has not.

If \(f(x, t)\) satisfies the Laplace equation \(f_{xx} + f_{tt} = 0\) and simultaneously the wave equation \(f_{xx} = f_{tt}\), then \(f(x, t) = ax + bt + c\).

Every smooth function satisfies the partial differential equation \(f_{xxyy} = f_{xyxy}\).

This follows from Clairot’s theorem.

The function \(f(x, y) = (x^4 - y^4)\) has neither a local maximum nor a local minimum at \((0, 0)\).

The function is both smaller and bigger than \(f(0, 0)\) for points near \((0, 0)\).

\(\int_0^1 \int_0^{\pi/2} r \, d\theta \, dr = \pi/4\).

This is the area of a quarter of the unit disc.
At a saddle point, the directional derivative is zero for two different vectors $u, v$.

**True.** The directional derivative can be both positive and negative at a saddle point. By the intermediate value theorem, there are two directions, where the directional derivative vanishes.

It is possible to find a function of two variables which has no maximum and no minimum.

**True.** There are many linear functions like that.

The value of the function $f(x, y) = e^x y$ at $(0.001, -0.001)$ can by linear approximation be estimated as $-0.001$.

**True.** Because the gradient at $(0, 0)$ is $(0, 1)$ and $f(0, 0) = 0$, the linear approximation is $L(x, y) = y$.

For any function $f(x, y, z)$ and any unit vectors $u, v$, one has the identity $D_u f(x, y, z) = D_u f(x, y, z) D_v f(x, y, z)$.

**False.** The directional derivative in the $u \times v$ direction has nothing to do with the directional derivatives into the other directions. An example, $u = (1, 0, 0); v = (0, 1, 0), f(x, y, z) = x + y$ is an example, where $D_u f(x, y, z) = 0$ but $D_u f = 1, D_v f = 1$.

Given 2 arbitrary points in the plane, there is a function $f(x, y)$ which has these points as critical points and no other critical points.

**True** If $(a, b), (c, d)$ are the two points, we want $\nabla f(x, y) = (x - a)(x - c), (y - b)(y - d))$. So, take $f(x, y) = x^3/3 - (a + c)x^2 + ax + y^3/3 - (b + d)y^2 + by$.

The maximum of $f(x, y)$ under the constraint $g(x, y) = 0$ is the same as the maximum of $g(x, y)$ under the constraint $f(x, y) = 0$.

**False** This can not be true, because the first problem is the same if we replace $g(x, y)$ with $2g(x, y)$, but this will change the value of the maximum of $g$ on the right hand side.

Assume $(x_0, y_0)$ is a critical point of $f(x, y)$ and $f_{xx} f_{yy} - f_{xy}^2 \neq 0$ at this point. Let $T$ be the tangent plane of the surface $S = \{ f(x, y) - z = 0 \}$ at $P = (x_0, y_0, f(x_0, y_0))$. If the intersection of $T$ with $S$ is a single point, then $(x_0, y_0)$ is a local max or local min.

**True.** The other possibility would be a saddle point, in which case, the tangent space intersects the surface in two curves which pass through the critical point.

The Key is

F F F T F F F F T T T T T F T F T
Problem 2) (30 points)

Match the parametric surfaces with their parameterization. No justification is needed.

<table>
<thead>
<tr>
<th>Enter I,II,III,IV here</th>
<th>Parameterization</th>
</tr>
</thead>
<tbody>
<tr>
<td>III</td>
<td>$(u, v) \mapsto (u \cos(v), u \sin(v), u^2 \cos(u)/(u + 1))$</td>
</tr>
<tr>
<td>II</td>
<td>$(u, v) \mapsto (u, v + u,</td>
</tr>
<tr>
<td>IV</td>
<td>$(u, v) \mapsto ((u - \sin(u)) \cos(v), (u - \cos(u)) \sin(v), u)$</td>
</tr>
<tr>
<td>I</td>
<td>$(u, v) \mapsto (u, v, u^2 - v^2)$</td>
</tr>
</tbody>
</table>
Problem 3) (40 points)

Find all the critical points of the function \( f(x, y) = xy(4 - x^2 - y^2) \). Are they maxima, minima or saddle points?

**Solution.** Taking derivatives of \( f(x, y) = 4xy - x^3y - xy^3 \) gives \( \nabla f(x, y) = (4y - 3x^2y - y^3, 4x - x^3 - 3xy^2) \). To solve the system

\[
\begin{align*}
y(4 - 3x^2 - y^2) &= 0 \quad (1) \\
x(4 - x^2 - 3y^2) &= 0 \quad (2)
\end{align*}
\]

We have the four following possibilities:

1) \( y = 0, x = 0 \)
2) \( 4 - 3x^2 - y^2 = 0, x = 0 \)
3) \( 4 - x^2 - 3y^2 = 0, y = 0 \)
4) \( 4 - 3x^2 - y^2 = 0, 4 - x^2 - 3y^2 = 0. \)

There are 9 critical points in total

1) gives the critical point \((0, 0)\).
2) gives the critical points \((0, 2), (0, -2)\).
3) gives the critical points \((2, 0), (-2, 0)\).
4) (subtract 3 times the second equation from the first): \((1, 1), (-1, 1), (1, 1), (1, -1)\).

The Hessian determinant (=discriminant) \( f_{xx}f_{yy} - f_{xy}^2 \) at a general point is \(-9(x^4 + y^4) - 16 + 24(x^2 + y^2) + 18x^2y^2 \) and \( f_{xx}(x, y) = -6xy \).

Applying the second derivative test gives

<table>
<thead>
<tr>
<th>Critical point</th>
<th>(-2, 0)</th>
<th>(-1, -1)</th>
<th>(-1, 1)</th>
<th>(0, 0)</th>
<th>(1, -1)</th>
<th>(1, 1)</th>
<th>(2, 0)</th>
<th>(-2, 0)</th>
<th>(2, 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discriminant</td>
<td>-64</td>
<td>32</td>
<td>32</td>
<td>-16</td>
<td>32</td>
<td>32</td>
<td>-64</td>
<td>-64</td>
<td>-64</td>
</tr>
<tr>
<td>( f_{xx} )</td>
<td>0</td>
<td>-6</td>
<td>6</td>
<td>0</td>
<td>-6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Analysis</td>
<td>saddle</td>
<td>max</td>
<td>min</td>
<td>saddle</td>
<td>min</td>
<td>min</td>
<td>max</td>
<td>saddle</td>
<td>saddle</td>
</tr>
</tbody>
</table>

Problem 4) (40 points)

Let \( f(x, y) = e^{(x-y)} \) so that \( f(\log(2), \log(2)) = 1 \). Find the equation for the tangent plane to the graph of \( f \) at \((\log(2), \log(2))\) and use it to estimate \( f(\log(2) + 0.1, \log(2) + 0.004) \).

**Solution.** The graph of \( f \) is a level curve of the function \( g(x, y, z) = z - f(x, y) \). The gradient at the point \((x_0, y_0, f(x_0, y_0) = (\log(2), \log(2), 1)\) is \((a, b, c) = (-1, 1, 1)\), so that the tangent plane has an equation \( ax + by + cz = -x + y + z = d \). and the constant \( d \) is obtained from \( d = -x_0 + y_0 + z_0 = 1 \). Therefore \( -x + y + z - 1 = 0 \).

At the point \((\log(2), \log(2), 1)\), the level surface \( g = 0 \) is close to the level surface \( L(x, y, z) = x - y - z + 1 = 0 \). If we plug in \( x = 0.1, y = 0.04, z = 0 \), we get 1.06.
Remark. We could have stayed in two dimensions and estimate $f(x_0 + dx, f(y_0 + dx)$ by $f(x_0, y_0) + (1, -1) \cdot (dx, dy) = 1 + dx - dy$ which is for $dx = 0.1, dy = 0.04$ equal to $1 + 0.1 - 0.04 = 1.06$.

Problem 5) (40 points)

$f$ is a function which depends on $x$ and $y$, where $x = u^3v$ and $y = u^2v^2$. When $(u, v) = (1, 1)$ $\frac{\partial f}{\partial x} = -5$, $\frac{\partial f}{\partial v} = 9$. What is $\frac{\partial f}{\partial u}$?

Solution. Apply the chain rule to $(u, v) \mapsto f(x(u, v), y(u, v)) = f(u^3v, u^2v^2)$:

$$f_u(u, v) = f_x(x(u, v), y(u, v))x_u(u, v) + f_y(x(u, v), y(u, v))y_u(u, v)$$

$$f_v(u, v) = f_x(x(u, v), y(u, v))x_v(u, v) + f_y(x(u, v), y(u, v))y_v(u, v)$$

Using $f_x(1, 1) = -5, f_v(1, 1) = 9, x_u(u, v) = 3, x_v(1, 1) = 1, y_u(1, 1) = 2, y_v(1, 1) = 2$ these two equations are

$$f_u(1, 1) = (-5)3 + f_y(1, 1)2$$

$$9 = (-5)1 + f_y(1, 1)2$$

The second equation can be solved for $f_y(1, 1) = 7$. Plugging this into the first equation gives $f_u(1, 1) = -1$.

Problem 6) (40 points)

A can is a cylinder with a circular base. Its surface area (top, bottom and sides) is $300\pi$ cm$^2$. What is the maximum possible volume of such a can?

Solution. We have the problem to extremize $f(r, h) = \pi r^2 h$ under the constraint $2\pi r^2 + 2\pi rh = 300\pi$. This is equivalent to extremize $f(r, h) = \pi r^2 h$ under the constraint $g(r, h) = r^2 + rh = 150$.

The Lagrange equations are

$$2\pi rh = \lambda(2r + h)$$

$$\pi r^2 = \lambda r$$

$$r^2 + rh = 150$$

The second equation gives $\pi r = \lambda$. Plugging in $\lambda$ into the first equation gives $h = 2r$. From the last equation, we get $r^2 + 2r^2 = 150$ or $r^2 = 50$. Therefore $r = 5\sqrt{2} cm, h = 10\sqrt{2} cm$. The maximal volume is $\pi r^2 h = 500\sqrt{2}$.
Problem 7) (40 points)

Evaluate \( \int_0^2 \int_0^{\sqrt{1-x^2}} \frac{xy^5}{x^2+y^2} \, dy \, dx \).

**Solution.** The integral is taken over the disc intersected with the first quadrant in the plane. In Polar coordinates, \( x^2 + y^2 = r, x = r \cos(\theta), y = r \sin(\theta) \) the integral is therefore

\[
\int_0^2 \int_0^{\pi/2} \frac{r \cos(\theta) r^5 \sin^5(\theta)}{r^2} \, r \, d\theta \, dr = \int_0^2 \int_0^{\pi/2} r^5 \cos(\theta) \sin^5(\theta) \, d\theta \, dr = 64/36 = 16/9
\]

Solution \(16/9\).

Problem 8) (40 points)

a) Find the area of the region \( D \) enclosed by the lines \( x = \pm 2 \) and the parabolas \( y = 1 + x^2 \), \( y = -1 - x^2 \).

b) Find the integral of \( f(x, y) = y^2 \) on the same region as in a). (The result can be interpreted as a moment of inertia).

**Solution.**

a) \( \int_{-2}^{2} \int_{-1-x^2}^{1+2x^2} 1 \, dy \, dx = \int_{-2}^{2} 2 + 2x^2 \, dx = 8 + 2x^3/3\big|_{-2}^{2} = 8 + 32/3 = 56/3 \).

b) \( \int_{-2}^{2} \int_{-1-x^2}^{1+2x^2} y^2 \, dy \, dx = \int_{-2}^{2} (1 + x^2)^3/3 \, dx = \int_{-2}^{2} (2 + 6x^2 + 6x^4 + 2x^6)/3 \, dx = 2216/35 \).