Math 21a Handout on Triple Integrals

The purpose of this handout is to provide a few more examples of triple integrals. In particular, I provide one example in the usual x-y-z coordinates, one in cylindrical coordinates and one in spherical coordinates.

**Example 1:** Here is the problem: Integrate the function \( f(x, y, z) = z \) over the tetrahedral pyramid in space where

- \( 0 \leq x. \)
- \( 0 \leq y. \)
- \( 0 \leq z. \)
- \( x + y + z \leq 1. \)

The integral in question is

\[
I = \iiint_R \left( \int_0^{1-x-y} z \, dz \right) \, dA,
\]

where \( R \) is the ‘shadow’ region in the x-y plane; the region where

- \( 0 \leq x, \)
- \( 0 \leq y, \)
- \( x + y \leq 1. \)

Indeed, a vertical line (where x and y are constant) will hit the pyramid only if x and y are non-negative and \( x + y \leq 1. \) Otherwise, one of the conditions in (1) is violated when \( z \geq 0. \) This line enters our pyramid from below where \( z = 0 \) and it then exits with z value where \( x + y + z = 1, \) which is to say where \( z = 1 - x - y. \) This information provides the lower bound to the z-integral, 0, and also the upper bound, \( 1 - x - y. \)

The integral over the z coordinate in (2) gives \( 2^{-1} (1 - x - y)^2, \) and so

\[
I = 2^{-1} \int_0^1 \left( \int_0^{1-x} (1 - x - y)^2 \, dy \right) \, dx.
\]

With regard to the upper and lower bounds to the x and y integrals in (4), remark that a vertical line (where x is constant) hits the shadow region \( R \) only if \( 0 \leq x \leq 1. \) Otherwise, one of the conditions in (3) will be violated when y is non-negative. (These bounds on x give the lower and upper bounds for the x integral in (4).) This line enters the shadow region where \( x = 0 \) and exits with y
value where $x + y = 1$. In this way, the lower bound to the $y$ integral in (4) is found to be 0 and the upper bound is found to be $1 - x$.

In any event, the $y$ integral in (4) gives $3^{-1} (1 - x)^3$ which leaves

$$I = 6^{-1} \int_0^1 (1 - x)^3 \, dx = (24)^{-1}.$$  

(5)

**Example 2:** The problem in this example is to integrate the function $z$ over the region where

- $0 \leq z$,
- $x^2 + y^2 \leq 1$,
- $z \leq 1 - x^2 - y^2$.

To accomplish this task, note that the integration region as just described involves only the coordinates $x$ and $y$ through the combination $x^2 + y^2$, and the function to be integrated doesn’t involve these coordinates at all. In particular, since $x$ and $y$ only appear here through $x^2 + y^2$, it makes sense to solve the problem using the cylindrical coordinates $(r, \theta, z)$. And, in terms of these coordinates, the integral in question has the form

$$I = zdz \, rdr \, d\theta.$$  

(6)

Here, the shadow region is seen to be the disk in the $x$-$y$ plane where $r \leq 1$ since a vertical line which hits the $x$-$y$ plane outside of this disk violates the middle point in (6). Meanwhile, the lower and upper bounds for the $z$-integral in (7) come about via the observation that a vertical line which has $r \leq 1$ enters the region where $z = 0$ (because of the first point in (6)) and exits where $z = 1 - r^2$ (because of the third point in (6)).

In any event, the $z$-integral in (7) gives $2^{-1} (1 - r^2)^2$ which makes

$$I = 2^{-1} \int_0^1 (1-r^2)^2 \left( \int_0^{2\pi} d\theta \right) rdr.$$  

(7)

Here, the bounds of 0 and 1 for the $r$ integral come from the fact that the shadow region is the disk in the $x$-$y$ plane where $r \leq 1$. This fact also explains why the 0 and $2\pi$ bounds for the $\theta$ integral.

The $\theta$ integral in (8) gives $2\pi$, which implies that

$$I = \pi \int_0^1 (1-r^2)^2 rdr.$$  

(8)
This last integral can be done by substituting $u = r^2$. The answer is $I = \pi/6$.

**Example 3:** The problem here is to integrate the function $f(x, y, z) = z$ over the upper half of the ball; this being the region where

- $0 \leq z$.
- $x^2 + y^2 + z^2 \leq 1$.  

Although this problem can be worked in cylindrical coordinates, spherical coordinates work as well. For the latter, the iterated integral is

$$I = \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 (\rho \cos \phi) \rho^2 \sin \phi \rho d\rho d\theta d\phi.$$  

Concerning (11), some explanation is in order: First, the term $\rho \cos \phi$ in the parenthesis is just the expression for $z$ in spherical coordinates. Second, the integration bounds of 0 and 1 for $\rho$, 0 and $2\pi$ for $\theta$ and 0 and $\pi/2$ for $\phi$ insure that the integration takes place only over the upper half of the ball. In particular, the function $z$ is non-negative only where $\phi \leq \pi/2$ and so the $\pi/2$ upper bound on the $\phi$ integral restricts the integration as required.

The $\rho$ integration in (11) can be done first with the result being $1/4$. The $\theta$ integration then provides a factor of $2\pi$. Finally, the $\phi$ integration provides a factor of $1/2$. (Use the substitution $u = \cos \phi$ while noticing that $du = -\sin \phi \, d\phi$.) Thus, $I = \pi/4$.  
