**Proof of Functional Equation by Contour Integral and Residues**

*Riemann's Zeta Function as Contour Integral.*

The Riemann zeta function is defined by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

for $z \in \mathbf{C}$ with $\operatorname{Re} z > 1$. We can write it as an integral as follows. For $\operatorname{Re} z > 1$ from

$$\int_{t=0}^{\infty} \frac{t^{z-1}}{e^t - 1} dt = \int_{t=0}^{\infty} \sum_{n=1}^{\infty} t^{z-1} e^{-nt} dt = \sum_{n=1}^{\infty} \frac{1}{n^z} \int_{t=0}^{\infty} t^{z-1} e^{-t} dt = \Gamma(z)\zeta(z),$$

it follows that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_{t=0}^{\infty} \frac{t^{s-1}}{e^t - 1} dt.$$

We now look at the meromorphic extension of $\zeta(z)$. Let $C$ be a contour which starts from positive infinity and goes toward the origin above the real axis and circles the origin once in the positive direction and then goes to positive infinity below the real axis. Then for $\operatorname{Re} z > 1$

$$\zeta(z) = \frac{-1}{2i \sin z\pi \, \Gamma(z)} \int_C \frac{(-w)^{z-1}}{e^w - 1} dw.$$

The precise evaluation of the integral is as follows.

$$(-w)^{z-1} = e^{(z-1)\log(-w)}$$

with

$$\log(-w) = \log \rho + i(\phi - \pi)$$

when $w = \rho e^{i\phi}$ and $0 < \phi < 2\pi$. So

$$\int_C \frac{(-w)^{z-1}}{e^w - 1} dw = \int_{\rho=\infty}^{0} \frac{e^{(z-1)(\log \rho - i\pi)}}{e^\rho - 1} d\rho + \int_{\rho=0}^{\infty} \frac{e^{(z-1)(\log \rho + i\pi)}}{e^\rho - 1} d\rho$$
$$= \left(e^{(z-1)\pi i} - e^{-(z-1)\pi i}\right) \int_{\rho=0}^{\infty} \frac{e^{(z-1)\log \rho}}{e^\rho - 1} d\rho$$

$$= 2i \sin\left((z-1)\pi\right) \int_{\rho=0}^{\infty} \frac{e^{(z-1)\log \rho}}{e^{\rho}-1} d\rho$$

$$= -2i \sin \pi z \int_{\rho=0}^{\infty} \frac{e^{(z-1)\log \rho}}{e^{\rho}-1} d\rho = -2i \sin \pi z \Gamma(z)\zeta(z).$$

Note that the condition $\operatorname{Re} z > 1$ is used to make sure that the integral around a small circle centered at the origin goes to zero as the radius of the circle goes to zero. When the condition $\operatorname{Re} z > 1$ is not assumed we can define the Riemann zeta function by

$$\zeta(z) = \frac{i\,\Gamma\left(1-z\right)}{2\pi} \int_C \frac{(-w)^{z-1}}{e^w - 1} dw$$

after we use the identity

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

*Pole-Set of Riemann Zeta Function.* We now look at the pole-set of $\zeta(z)$. The integral

$$\int_C \frac{(-w)^{z-1}}{e^w - 1} dw$$

is holomorphic at any value of $z \in \mathbb{C}$. The pole-set of $\Gamma\left(1-z\right)$ is located at the non-positive integral values of $1-z$ which means that the positive integral values of $z$. However, we know that $\zeta(z)$ is holomorphic for $\operatorname{Re} z > 1$. Hence the only possible pole of $\zeta(z)$ is a simple pole at $z = 1$. Since

$$\int_C \frac{1}{e^w - 1} dw = 2\pi i$$

and

$$\Gamma\left(1-z\right) = -\frac{1}{z-1} + \cdots,$$

it follows that the residue of $\zeta(z)$ at $z = 1$ is precisely 1. By using

$$\frac{a}{e^z - 1} = 1 - \frac{1}{2}z + \sum_{n=1} (-1)^{n-1} B_n \frac{z^{2n}}{(2n)!},$$

where $B_n$ is the $n$-th Bernoulli number, we conclude that

$$\zeta(0) = -\frac{1}{2}, \quad \zeta\left(-2m\right) = 0,$$

$$\zeta\left(1-2m\right) = \frac{\left(-1\right)^m B_m}{2m},$$

for any positive integer $m$.

Recall that

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42},$$
$$B_n = \frac{(2n)!}{2^{2n-1}\,\pi^{2n}} \sum_{\ell=1}^{\infty} \frac{1}{\ell^{2n}}.$$

*Derivation of Functional Equation.* We now look at the functional equation for the Riemann zeta function.

$$\zeta(1-z) = 2^{1-z}\pi^{-z}\cos\frac{\pi z}{2}\,\Gamma(z)\zeta(z).$$

To prove this, we use

$$\zeta(z) = \frac{-1}{2i\sin z\pi\,\Gamma(z)} \int_C \frac{(-w)^{z-1}}{e^w - 1} dw$$

and deform the contour to $C_n$ which starts with positive infinity goes above the real axis to $(2n+1)\pi$ and then goes counter-clockwise along the rectangle with vertices at $(\pm 1 \pm i)\,(2n+1)\pi$ once to get back to $(2n+1)\pi$ and then goes below the real axis to positive infinity. The limit contour integral is zero. When we pass from $C$ to $C_n$ as $n \to \infty$, we pass over the poles of

$$\frac{(-w)^{z-1}dw}{e^w - 1}$$

at $2i\,(\mathbf{Z} - 0)$. The residue of

$$\frac{(-w)^{z-1}dw}{e^w - 1}$$

at $2mi\pi$ for $m > 0$ is

$$e^{(z-1)\left(\log 2m\pi - \frac{1}{2}i\pi\right)} = (2m\pi)^{z-1}\, ie^{-\frac{1}{2}i\pi z}.$$

The residue at $-2mi\pi$ is

$$e^{(z-1)\left(\log 2m\pi + \frac{1}{2}i\pi\right)} = -\,(2m\pi)^{z-1}\, ie^{\frac{1}{2}i\pi z}.$$

Their sum is

$$(2m\pi)^{z-1}\, 2\sin\frac{1}{2}\pi z.$$

Hence

$$
\begin{aligned}
-2i\sin z\pi\Gamma(z)\zeta(z) &= -2\pi i\sum_{m=1}^{\infty}(2m\pi)^{z-1}\,2\sin\frac{1}{2}\pi z \\
&= -2\pi i\,(2\pi)^{z-1}\,\zeta(1-z)2\sin\frac{1}{2}\pi z
\end{aligned}
$$

for $z$ with $\operatorname{Re} z < 0$, from which it follows that

$$\zeta(1-z) = 2^{1-z}\pi^{-z}\cos\frac{\pi z}{2}\,\Gamma(z)\zeta(z)$$

for all $z \in \mathbf{C}$. From

$$\Gamma(s) = 2^{s-1}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{1}{2}}$$

it follows that, when we define

$$\xi(s) = \frac{1}{2}s\,(s-1)\,\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s),$$

the functional equation simply reads $\xi(1-s) = \xi(s)$.