The Assignment Problem
and
the Hungarian Method

Example 1: You work as a sales manager for a toy manufacturer, and you currently have three salespeople on the road meeting buyers. Your salespeople are in Austin, TX; Boston, MA; and Chicago, IL. You want them to fly to three other cities: Denver, CO; Edmonton, Alberta; and Fargo, ND. The table below shows the cost of airplane tickets in dollars between these cities.

From \ To	Denver	Edmonton	Fargo
Austin	250	400	350
Boston	400	600	350
Chicago	200	400	250

Where should you send each of your salespeople in order to minimize airfare?

We can represent the table above as a *cost matrix*.

$$\begin{bmatrix} 250 & 400 & 350 \\ 400 & 600 & 350 \\ 200 & 400 & 250 \end{bmatrix}$$

Let's look at one possible assignment.

$$\begin{bmatrix} \boxed{250} & 400 & 350 \\ 400 & \boxed{600} & 350 \\ 200 & 400 & \boxed{250} \end{bmatrix}$$

The total cost of this assignment is

$\$250 + \$600 + \$250 = \$1100.$

Here's another possible assignment.

$$\begin{bmatrix} \boxed{250} & 400 & 350 \\ 400 & 600 & \boxed{350} \\ 200 & \boxed{400} & 250 \end{bmatrix}$$

The total cost of this assignment is
$\$250 + \$350 + \$400 = \$1000$.

After checking all six possible assignments we can determine that the optimal one is the following.

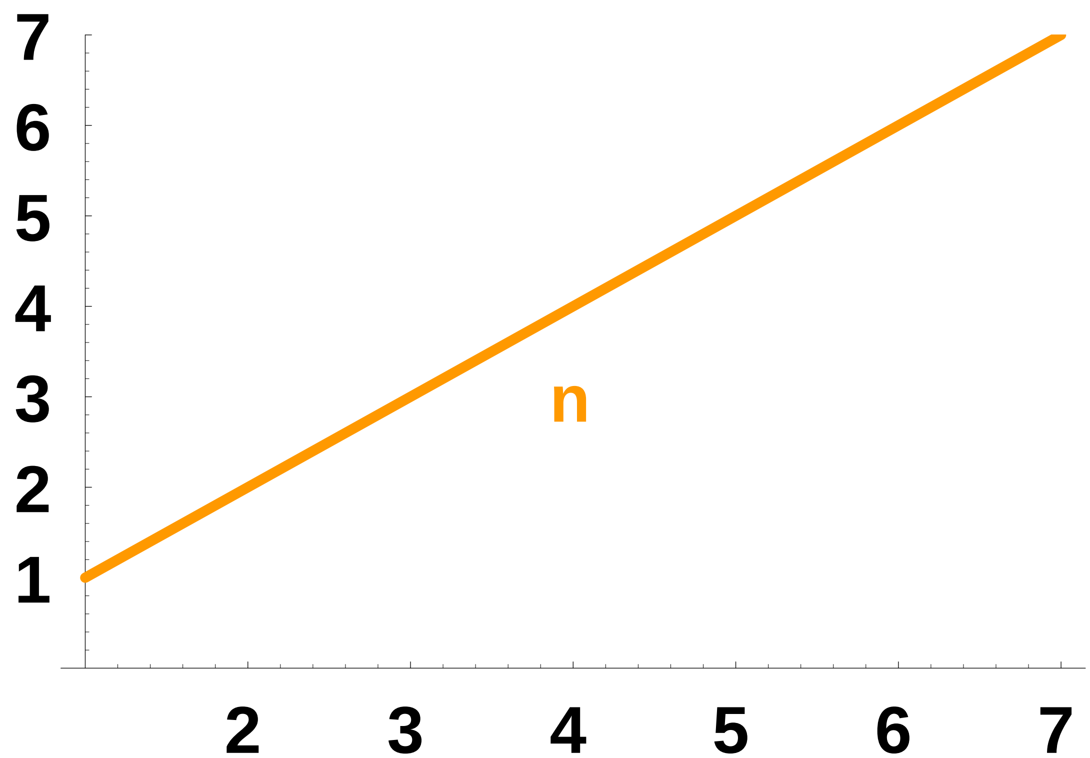
$$\begin{bmatrix} 250 & \boxed{400} & 350 \\ 400 & 600 & \boxed{350} \\ \boxed{200} & 400 & 250 \end{bmatrix}$$

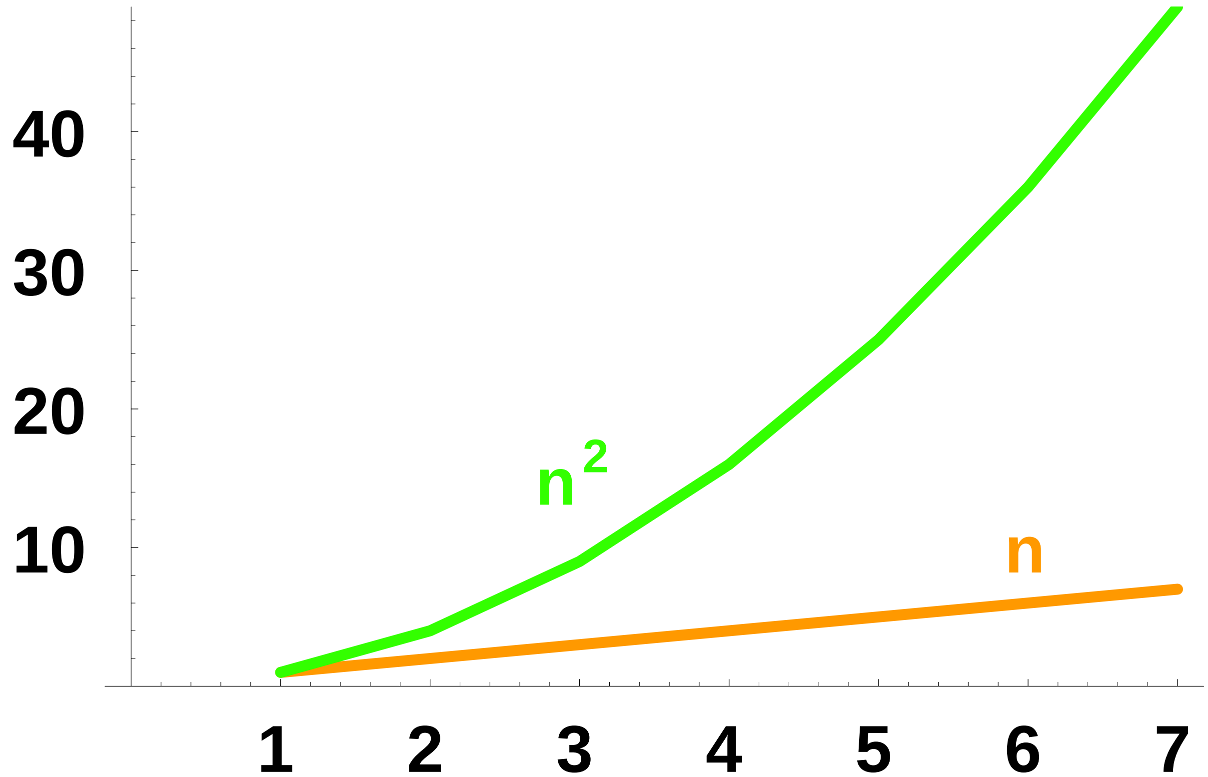
The total cost of this assignment is
$\$400 + \$350 + \$200 = \$950.$

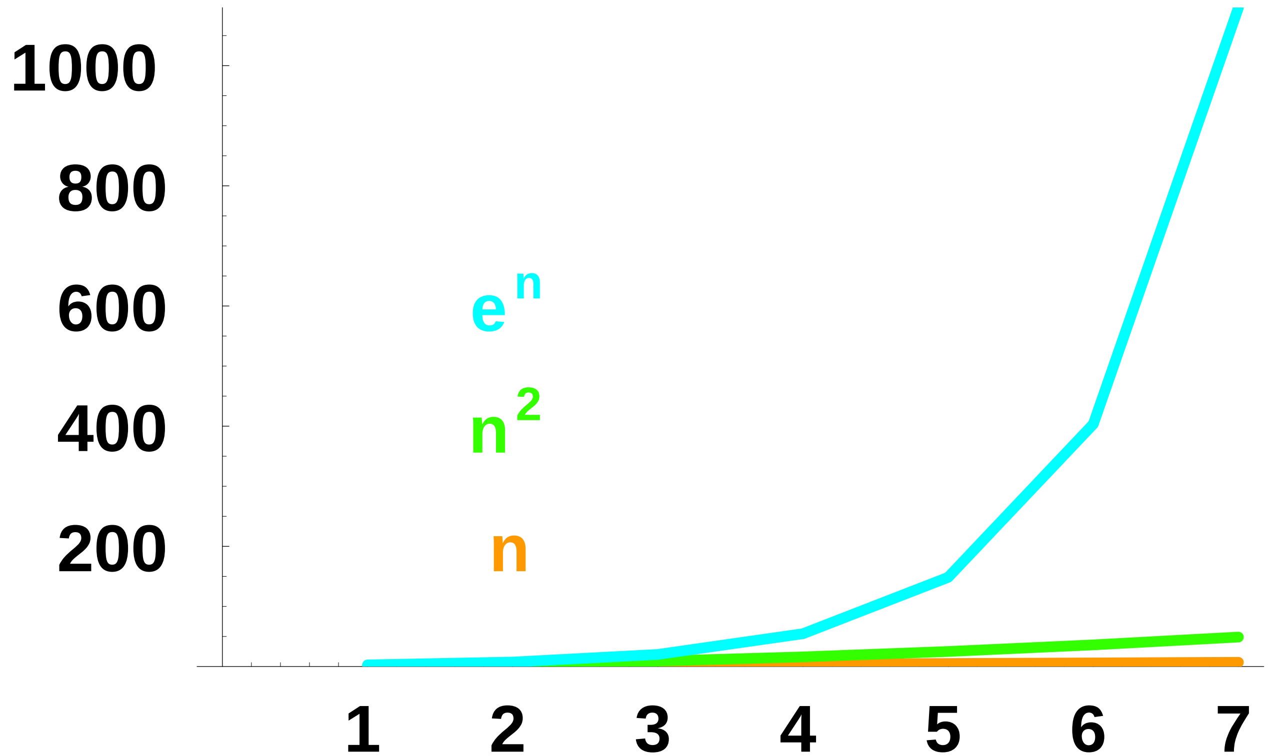
Thus your salespeople should travel from Austin to Edmonton, Boston to Fargo, and Chicago to Denver.

Trial and error works well enough for this problem, but suppose you had ten salespeople flying to ten cities? How many trials would this take?

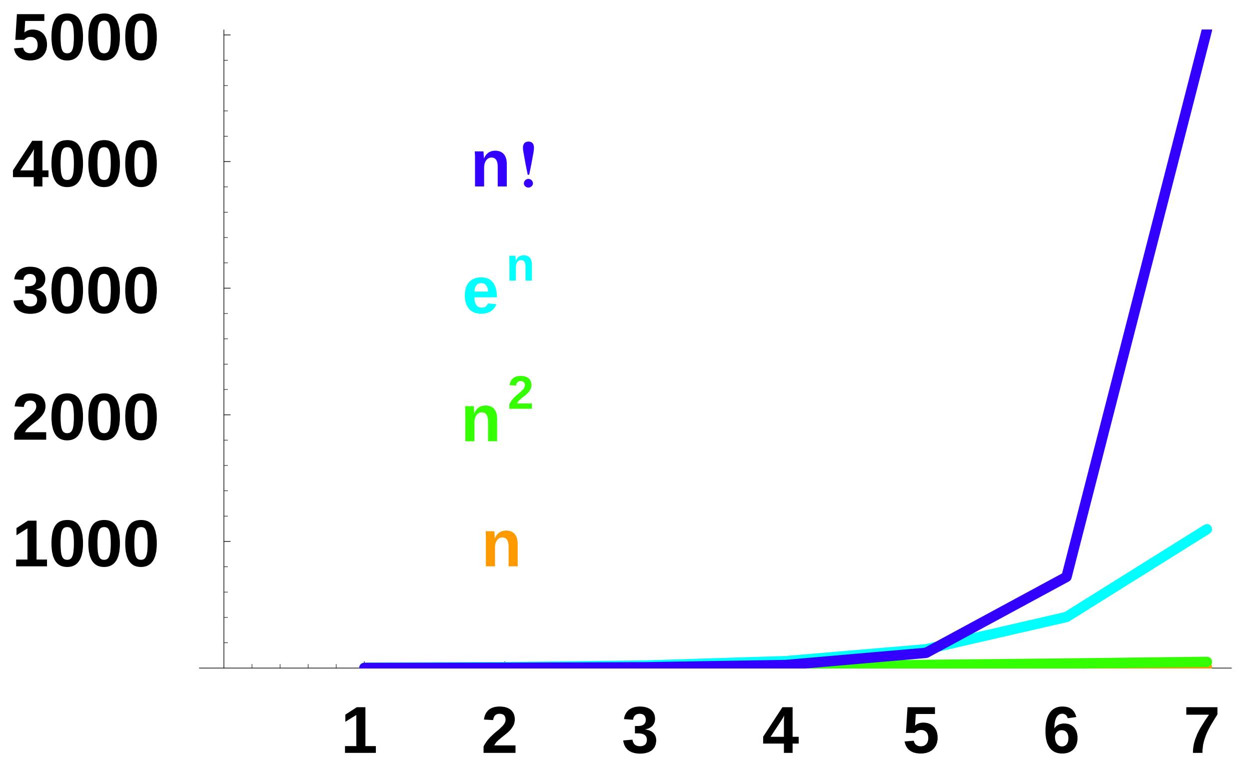
There are $n!$ ways of assigning n resources to n tasks. That means that as n gets large, we have too many trials to consider.


7
6
5
4
3
2
1
n
2
3
4
5
6
7


40
30
20
10
n2
n
1
2
3
4
5
6
7



1000
800
600
400
200
1 2 3 4 5 6 7
e^n
n^2
n


5000
4000
3000
2000
1000
n!
e^n
n^2
n
1
2
3
4
5
6
7

Theorem: If a number is added to or subtracted from all of the entries of any one row or column of a cost matrix, then on optimal assignment for the resulting cost matrix is also an optimal assignment for the original cost matrix.

The Hungarian Method: The following algorithm applies the above theorem to a given $n \times n$ cost matrix to find an optimal assignment.

Step 1. Subtract the smallest entry in each row from all the entries of its row.

Step 2. Subtract the smallest entry in each column from all the entries of its column.

Step 3. Draw lines through appropriate rows and columns so that all the zero entries of the cost matrix are covered and the *minimum* number of such lines is used.

Step 4. *Test for Optimality:* (i) If the minimum number of covering lines is n, an optimal assignment of zeros is possible and we are finished. (ii) If the minimum number of covering lines is less than n, an optimal assignment of zeros is not yet possible. In that case, proceed to Step 5.

Step 5. Determine the smallest entry not covered by any line. Subtract this entry from each uncovered row, and then add it to each covered column. Return to Step 3.

Example 1: You work as a sales manager for a toy manufacturer, and you currently have three salespeople on the road meeting buyers. Your salespeople are in Austin, TX; Boston, MA; and Chicago, IL. You want them to fly to three other cities: Denver, CO; Edmonton, Alberta; and Fargo, ND. The table below shows the cost of airplane tickets in dollars between these cities.

From \ To	Denver	Edmonton	Fargo
Austin	250	400	350
Boston	400	600	350
Chicago	200	400	250

Where should you send each of your salespeople in order to minimize airfare?

Step 1. Subtract 250 from Row 1, 350 from Row 2, and 200 from Row 3.

$$\begin{bmatrix} 250 & 400 & 350 \\ 400 & 600 & 350 \\ 200 & 400 & 250 \end{bmatrix} \sim \begin{bmatrix} 0 & 150 & 100 \\ 50 & 250 & 0 \\ 0 & 200 & 50 \end{bmatrix}$$

Step 2. Subtract 0 from Column 1, 150 from Column 2, and 0 from Column 3.

$$\begin{bmatrix} 0 & 150 & 100 \\ 50 & 250 & 0 \\ 0 & 200 & 50 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 100 \\ 50 & 100 & 0 \\ 0 & 50 & 50 \end{bmatrix}$$

Step 3. Cover all the zeros of the matrix with the minimum number of horizontal or vertical lines.

$$\begin{bmatrix} 0 & 0 & 100 \\ 50 & 100 & 0 \\ 0 & 50 & 50 \end{bmatrix}$$

Step 4. Since the minimal number of lines is 3, an optimal assignment of zeros is possible and we are finished.

Since the total cost for this assignment is 0, it must be an optimal assignment.

$$\begin{bmatrix} 0 & \boxed{0} & 100 \\ 50 & 100 & \boxed{0} \\ \boxed{0} & 50 & 50 \end{bmatrix}$$

Here is the same assignment made to the original cost matrix.

$$\begin{bmatrix} 250 & \boxed{400} & 350 \\ 400 & 600 & \boxed{350} \\ \boxed{200} & 400 & 250 \end{bmatrix}$$

Example 2: A construction company has four large bulldozers located at four different garages. The bulldozers are to be moved to four different construction sites. The distances in miles between the bulldozers and the construction sites are given below.

Bulldozer \ Site	A	B	C	D
1	90	75	75	80
2	35	85	55	65
3	125	95	90	105
4	45	110	95	115

How should the bulldozers be moved to the construction sites in order to minimize the total distance traveled?

Step 1. Subtract 75 from Row 1, 35 from Row 2, 90 from Row 3, and 45 from Row 4.

$$\begin{bmatrix} 90 & \color{red}{75} & 75 & 80 \\ \color{red}{35} & 85 & 55 & 65 \\ 125 & 95 & \color{red}{90} & 105 \\ \color{red}{45} & 110 & 95 & 115 \end{bmatrix} \sim \begin{bmatrix} 15 & 0 & 0 & 5 \\ 0 & 50 & 20 & 30 \\ 35 & 5 & 0 & 15 \\ 0 & 65 & 50 & 70 \end{bmatrix}$$

Step 2. Subtract 0 from Column 1, 0 from Colum 2, 0 from Column 3, and 5 from Column 4.

$$\begin{bmatrix} 15 & 0 & 0 & 5 \\ 0 & 50 & 20 & 30 \\ 35 & 5 & 0 & 15 \\ 0 & 65 & 50 & 70 \end{bmatrix} \sim \begin{bmatrix} 15 & 0 & 0 & 0 \\ 0 & 50 & 20 & 25 \\ 35 & 5 & 0 & 10 \\ 0 & 65 & 50 & 65 \end{bmatrix}$$

Step 3. Cover all the zeros of the matrix with the minimum number of horizontal or vertical lines.

$$\begin{bmatrix} 15 & 0 & 0 & 0 \\ 0 & 50 & 20 & 25 \\ 35 & 5 & 0 & 10 \\ 0 & 65 & 50 & 65 \end{bmatrix}$$

Step 4. Since the minimal number of lines is less than 4, we have to proceed to Step 5.

Step 5. Note that 5 is the smallest entry not covered by any line. Subtract 5 from each uncovered row.

$$\begin{bmatrix} 15 & 0 & 0 & 0 \\ \color{blue}{0} & \color{blue}{50} & \color{blue}{20} & \color{blue}{25} \\ \color{blue}{35} & \color{blue}{5} & \color{blue}{0} & \color{blue}{10} \\ \color{blue}{0} & \color{blue}{65} & \color{blue}{50} & \color{blue}{65} \end{bmatrix} \sim \begin{bmatrix} 15 & 0 & 0 & 0 \\ -5 & 45 & 15 & 20 \\ 30 & 0 & -5 & 5 \\ -5 & 60 & 45 & 60 \end{bmatrix}$$

Now add 5 to each covered column.

$$\begin{bmatrix} \color{blue}{15} & 0 & \color{blue}{0} & 0 \\ \color{blue}{-5} & 45 & \color{blue}{15} & 20 \\ \color{blue}{30} & 0 & \color{blue}{-5} & 5 \\ \color{blue}{-5} & 60 & \color{blue}{45} & 60 \end{bmatrix} \sim \begin{bmatrix} 20 & 0 & 5 & 0 \\ 0 & 45 & 20 & 20 \\ 35 & 0 & 0 & 5 \\ 0 & 60 & 50 & 60 \end{bmatrix}$$

Now return to Step 3.

Step 3. Cover all the zeros of the matrix with the minimum number of horizontal or vertical lines.

$$\begin{bmatrix} 20 & 0 & 5 & 0 \\ 0 & 45 & 20 & 20 \\ 35 & 0 & 0 & 5 \\ 0 & 60 & 50 & 60 \end{bmatrix}$$

Step 4. Since the minimal number of lines is less than 4, we have to return to Step 5.

Step 5. Note that 20 is the smallest entry not covered by a line. Subtract 20 from each uncovered row.

$$\begin{bmatrix} 20 & 0 & 5 & 0 \\ 0 & 45 & 20 & 20 \\ 35 & 0 & 0 & 5 \\ 0 & 60 & 50 & 60 \end{bmatrix} \sim \begin{bmatrix} 20 & 0 & 5 & 0 \\ -20 & 25 & 0 & 0 \\ 35 & 0 & 0 & 5 \\ -20 & 40 & 30 & 40 \end{bmatrix}$$

Then add 20 to each covered column.

$$\begin{bmatrix} 20 & 0 & 5 & 0 \\ -20 & 25 & 0 & 0 \\ 35 & 0 & 0 & 5 \\ -20 & 40 & 30 & 40 \end{bmatrix} \sim \begin{bmatrix} 40 & 0 & 5 & 0 \\ 0 & 25 & 0 & 0 \\ 55 & 0 & 0 & 5 \\ 0 & 40 & 30 & 40 \end{bmatrix}$$

Now return to Step 3.

Step 3. Cover all the zeros of the matrix with the minimum number of horizontal or vertical lines.

$$\begin{bmatrix} 40 & 0 & 5 & 0 \\ 0 & 25 & 0 & 0 \\ 55 & 0 & 0 & 5 \\ 0 & 40 & 30 & 40 \end{bmatrix}$$

Step 4. Since the minimal number of lines is 4, an optimal assignment of zeros is possible and we are finished.

$$\begin{bmatrix} 40 & 0 & 5 & \boxed{0} \\ 0 & 25 & \boxed{0} & 0 \\ 55 & \boxed{0} & 0 & 5 \\ \boxed{0} & 40 & 30 & 40 \end{bmatrix}$$

Since the total cost for this assignment is 0, it must be an optimal assignment.

Here is the same assignment made to the original cost matrix.

$$\begin{bmatrix} 90 & 75 & 75 & \boxed{80} \\ 35 & 85 & \boxed{55} & 65 \\ 125 & \boxed{95} & 90 & 105 \\ \boxed{45} & 110 & 95 & 115 \end{bmatrix}$$

So we should send Bulldozer 1 to Site D, Bulldozer 2 to Site C, Bulldozer 3 to Site B, and Bulldozer 4 to Site A.

Summary

The Assignment Problem: Suppose we have n resources to which we want to assign to n tasks on a one-to-one basis. Suppose also that we know the cost of assigning a given resource to a given task. We wish to find an optimal assignment–one which minimizes total cost.

The Mathematical Model: Let $c_{i,j}$ be the cost of assigning the ith resource to the jth task. We define the *cost matrix* to be the $n \times n$ matrix

$$C = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{bmatrix}.$$

An *assignment* is a set of n entry positions in the cost matrix, no two of which lie in the same row or column. The sum of the n entries of an assignment is its *cost*. An assignment with the smallest possible cost is called an *optimal assignment*.

The Hungarian Method: The *Hungarian method* (see above) is an algorithm which finds an optimal assignment for a given cost matrix.