VLAOSV DYNAMICS (*)

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ABSTRACT. Vlasov dynamics generalizes n-body particle dynamics. But there is a geometric twist.

VLAOSV DYNAMICS. Let M be a manifold of dimension p with measure m and cotangent bundle T *M and let N be a manifold of dimension q with cotangent bundle T *N. Take N = R, T *M = R^n. A one-parameter family of maps X^t = (f^t, g^t): T *M → T *N is defined by the differential equation

\[ \dot{f} = g, \dot{g} = -\int_{T^*N} \nabla V(f(\omega) - f(\omega))) \, d\omega \]

where V is a potential. This is a Hamiltonian system.

FACT. If (f, g) move according to \( \dot{f} = g, \dot{g} = -\int \nabla V(f(\omega) - f(\omega))) \, d\omega \), then the density \( P = (f, g) \) satisfies the Vlasov equation

\[ \frac{d}{dt} P(x, y, t) + y \nabla_x P(x, y, t) - E(x) \nabla_y P(x, y, t) = 0 \]

with \( E(x) = \int_{T^*N} \nabla_x V(x - x') \, dx'dx \).

PROOF. We have \( \int \nabla \cdot (\frac{P}{V} \nabla V(x')) \, dxdy = \frac{1}{2} \int_{T^*N} \nabla_x h(x, y)P'(x, y) \, dxdy = \frac{1}{2} \int_{T^*N} \nabla_x h(x, y)P'(x, y) \, dxdy = \frac{1}{2} \int_{T^*N} \nabla_x h(x, y)P'(x, y) \, dxdy = 0 \).

EXAMPLES. 1) If \( T^*M \) is zero dimension with \( n \) points \( \{x_1, \ldots, x_n\} \), then \( X^t \) describes the evolution of \( n \) particles \( (f^t, g^t) = (X^t, \omega) \). Vlasov dynamics is therefore a generalisation of n-body dynamics. 2) If \( N = M \), then \( X^t \) are volume-preserving deformations of \( T^*M \).

LINEARIZED MOTION. The evolution of \( DX^t \) at a point \( \omega \in M \) is \( DF_\omega(f) = -\int_{T^*N} \nabla_x V(f(\omega) - f(\omega))) \, d\omega \). The critical points of \( Df \) can only appear for \( \omega \), where \( Df(\omega) \) is linearly independent. More generally \( Y_0(t) = \omega \in T^*M \mid DX_0(\omega) \) has rank \( 2q - k = dim(T^*N) - k \) is time independent. The set \( Y_0 \) contains \( \{\omega \mid Df(\omega) = \lambda \} \).

LYAPUNOV EXPONENT.

\[ \lambda(\omega) = \lim sup_{t \to \infty} t^{-1} \log \|D(X^t(\omega))\| \in [0, \infty]. \]

\( \lambda(\omega) \) is the maximal Lyapunov exponent of the SL(2q, R)-cocycle \( A = A(f) \) along an orbit \( (f^t, g^t) \). The Lyapunov exponent could be infinite.

HESSIAN. Differentiation of \( DF = Df(f^t) \) at a critical point \( \omega^* \) gives \( Df^t(f^t) = B(f^t)Df^t(f^t) \). The eigenvalues \( \lambda_i \) of the Hessian \( D^2f \) satisfy \( \lambda_i = \lambda_i(f) \lambda_j \).

EQUILIBRIUM MEASURES. Equilibrium measures are stationary solutions of the Vlasov equation. One can get them with a Maxwellian ansatz \( P(x, y) = C \exp(-\beta V(x, y)) \). For \( q = 0 \), they are called Bernstein-Greens-Kraskul (BGK) modes.

If \( q = 1 \), they satisfy the integro-differential equation \( Q\omega = -\beta S(\omega) \). Then the Maxwellian distribution \( P(x, y) = S(y)Q(x) \) is an equilibrium solution of the Vlasov equation to the potential V because \( y_0 \nabla P = y \frac{S(y)}{Q(x)} \). They are called Bernstein-Greens-Kraskul (BGK) modes.

\[ \frac{d}{dt} \int_{T^*N} \nabla_x V(x - x') \, dx'dy = \int_{T^*N} \nabla_x V(x - x') \, dx'dy = \int_{T^*N} \nabla_x V(x - x') \, dx'dy \]

which holds in a neighborhood chart of \( f \). The standard Piccard existence theorem for differential equations in Banach manifolds assures local existence.

The global Lipschitz assumption and a Gronwall estimate assures that \( \|X(\omega)\| \) cannot grow faster than exponentially leading to global existence.

The result could also be derived from the existence theorem applied to finite measure where the evolution is a n-body evolution. Uniqueness and global existence of solutions on a dense set of point measures implies uniqueness in general if the dynamics depends continuously on the measure m.

BATT-NEUZNSRT-BROWN-HEPP-DOBRUSHIN EXISTENCE THEOREM.

If \( \nabla_x V \) is bounded and globally Lipschitz continuous, then \( f(\omega) = -\int \nabla_x V(f(\omega) - f(\omega'))) \, d\omega(\omega') \) has a unique global solution. Consequently the Vlasov equation has a unique and smooth solution. If \( V \) and \( P \) are smooth, then \( P^t \) is piecewise smooth.

PROOF. Take \( M = T^*N \) and let \( m = P^t \) be the initial measure. The Hamiltonian differential equation for \( X = (f, g) \) on the complete metric space of all continuous maps from \( M \) to \( T^*N \), which is a Banach manifold over the Banach manifold \( C^\infty(M, T^*N) \). The distance is \( d(h, h') = sup_{h \in M} d(h, h') \).

With \( X^t = I_\omega, \) the initial data \( (f_0, g_0)(x) = x \), we have \( P_0 = (f_0, g_0) \). The differential equation \( \dot{f} = g \) and \( \dot{g} = G(f) = -\int_{T^*N} \nabla_x V(f(\omega) - f(\omega))) \, d\omega(\omega') \) is \( C(M, T^*N) \) has a unique solution: because of Lipshitz continuity

\[ ||G(f) - G(f')||_\omega \leq 2\|D(\nabla_x V)||_\omega \cdot ||f - f'||_\omega \]

which holds in a neighborhood chart of \( f \). The standard Piccard existence theorem for differential equations in Banach manifolds assures local existence.