THEOREM OF DEVANEY-NITECKI. Fix $b \neq 0$. For large enough $c$, the Henon map $H : (x, y) \mapsto (x^2 - c - by, x)$ has an invariant set $K$ such that $T$ restricted to $K$ is conjugated to the shift

$$S = \ldots, x_{-1}, x_0, x_1, x_2, \ldots \mapsto \ldots, x_{0}, x_1, x_2, x_3, \ldots$$
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on all sequences with two symbols.

PROOF. With the new parameter $a = 1/\sqrt{c}$ and the new coordinates $q = x \cdot a, p = y \cdot a$, the map becomes

$$T(q, p) \mapsto \left(\frac{q^2}{a^2} - 1 - bp, q\right)$$

and is equivalent to the recurrence

$$a \cdot q_{n+1} + a \cdot b \cdot q_{n-1} = q_n^2 - 1.$$

We look for sequences $q_n = q(S^n x)$, where $S$ is the shift on the space of all sequence: $X = \{-1, 1\}^\mathbb{Z}$ and where $q$ is a continuous map from $X$ to $R$. We have to solve

$$a \cdot q(Sx) + a \cdot b \cdot q(S^{-1} x) - (q(x)^2 - 1) = 0.$$

With the map $F : R \times C(X) \rightarrow C(X)$ defined by

$$F(a, q)(x) = a \cdot q(Sx) + a \cdot b \cdot q(S^{-1} x) - (q(x)^2 - 1)$$

this equation can be rewritten as $F(a, q) = 0$. The partial derivative $F_q(a, q)$ is

$$F_q(a, q)u = a(u(Sx) + b \cdot u(S^{-1} x)) - 2q \cdot u.$$

The map $F(0, q) : C(X) \rightarrow C(X)$ has the property that every function $q \in C(X)$ with values in $\{-1, 1\}$ is a solution of $F(0, q) = 0$. We take for such a solution the map $q(x) = x_0$.

The derivative $F_q(0, q)$ is the linear map

$$(F_q(0, q)u) = -2q \cdot u$$

which is invertible because $q$ is bounded away from 0.

By the implicit function theorem, there exists a solution $a \rightarrow q_n = G(a)$ satisfying $F(a, q_n) = 0$ for small $a$. Define $\phi_a : X \rightarrow R^2$ by

$$\phi_a(x) = (q(x), q(S^{-1} x)).$$

The map $\phi_a$ is continuous, because $q$ and $T$ are continuous.

Using $F(a, q) = 0$, we check that

$$\phi_a \circ T(x) = \left(q(Sx), q(x)\right) = \frac{(q(x)^2 - 1)}{a} - b \cdot q(S^{-1} x), q(x))$$

$$= T(q(x), q(S^{-1} x)) = T \circ \phi_a(x)$$

for all $x \in X$.

The map is injective because if two points $x, y$ are mapped to the same point in $R^2$ then the fact that $q_n(x) = q_n(y)$ implies $x_0 = y_0$. The conjugation $\phi_a \circ S^g(x) = T^g \circ \phi_a(x)$ gives us $T^n(x) = T^n(y)$ and so $x_n = y_n$ for all $n$.

$\phi$ has a continuous inverse because every bijective map from a compact space to a compact space has a continuous inverse. The map is indeed a homeomorphism from $X$ to a closed subset $K = \phi(X) \subset R^2$.

HORSE SHOES IN THE STANDARD MAP. For large enough $c$, the Standard map $T : (x, y) \mapsto (2x + \cos(x) - y, x)$ has an invariant set $K$ such that $T$ restricted to $K$ is conjugated to the shift

$$S = \ldots, x_{-1}, x_0, x_1, x_2, \ldots \mapsto \ldots, x_{0}, x_1, x_2, x_3, \ldots$$

on all sequences with two symbols.

PROOF. If $T^n(q, p) = (q_n, p_n)$ is an orbit of the Standard map, then $p_n = q_{n+1}$ and so $q_{n+1} - 2q_n + q_{n-1} + \cos(x_n) = 0$. With $c = 1/\epsilon$, this means

$$\epsilon(q_{n+1} - 2q_n + q_{n-1}) + \sin(x_n) = 0$$

Let $X$ be all $\{0, 1\}$ sequences. Consider the space of all continuous functions $q$ from $X$ to $[0, 2\pi]$. If we find a solution $q$ to the equation

$$F(c, q) = \epsilon(q(x^2) - 2q(x) + q((x^2)^{-1})) + \sin(q(x)) = 0$$

then $q$ is a conjugation from $(X, \sigma)$ to $(q(X), T)$ showing that we can find a shift similar as the horse shoe construction does.

(i) There is a solution for $\epsilon = 0$. Just take $q(x) = \pi x_0$. Because $\sin(0) = \sin(\pi) = 0$, the equation $\sin(q(x)) = 0$ is satisfied.

(ii) In order to have a solution for small $\epsilon$, we compute the derivative of $L = F_c(0, q) = \cos(q)$ and see whether it is invertible. Indeed, since $L = \cos(q(x)) = \pm 1$, we can invert $L$, the inverse is actually equal to $L$. (Note that $F$ has as an argument a function $q$ and the the derivative $F_q(a, q) = \lim_{x \rightarrow 0} (F(a, q + u) - F(a, q))/u$ is defined with respect to the function $q$. It was computed in the same way as derivatives with respect to real parameters.)

(iii) The implicit function theorem now assures that we can find such a function $q$, which satisfies $F(q, x) = 0$. This function $q$ conjugates the shift with the standard map $T$, restricted to the set $K = \phi(X)$.

JULIA SETS. The same construction works also for the map $f(z) = a(z^2 - 1)$. We look for a function $q \in C(X, C)$ such that $q(x) - a(x^2 + 1) = 0$. With $\epsilon = 1/\alpha$, this is $F(\epsilon, q) = \epsilon q(x) - (z^2 - 1) = 0$.

For $\epsilon = 0$, the function $q(x) = (2x_0 - 1)$ is a solution. The derivative $L = F_c(0, q) = 2q$ is invertible. We have solutions for small $\epsilon$, which corresponds to large $a$. Actually, the image $q(X)$ is just the Julia set of $f$.

SUMMARY. The anti-integrable limit construction allows to get embedded shifts in a purely analytic way using the implicit function theorem. In comparison, the construction of a horse shoe is a geometric construction. Finding a generating partition is a more combinatorial task. The shift brings different areas of mathematics together.