ABSTRACT. This is an overview over the stability of equilibrium points of linear differential equations in the plane.

LINEAR SYSTEMS. A linear differential equation in two dimensions has the form
\[ \frac{dx(t)}{dt} = ax + by, \]
\[ \frac{dy(t)}{dt} = cx + cy. \]

It can be written as \( \frac{d}{dt} \vec{x}(t) = A \vec{x}(t) \) with a vector \( \vec{x} \) and a matrix \( A \). We denote the eigenvalues of \( A \) with \( \lambda_1 \) and \( \lambda_2 \).

If the eigenvalues are different, one can diagonalize \( A \). In the eigenbasis of \( A \), the matrix is \( B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \)
and the differential equation becomes
\[ \frac{dx(t)}{dt} = \lambda_1 x, \]
\[ \frac{dy(t)}{dt} = \lambda_2 y. \]

with explicit solution \( x(t) = e^{\lambda_1 t} x(0), y(t) = e^{\lambda_2 t} y(0) \).

PHASE-PORTRATES. We plot some vector fields and typical orbits

AREA PRESERVATION. A differential equation for which we have solutions for all times defines for each \( t \) a map \( T_t \) in the plane.

We say a differential equation \( \frac{dx}{dt} = F(x) \) is area-preserving if each of the time \( t \) maps \( T_t \) is area preserving.

DIVERGENCE. If \( F \) is a vector field, we denote by \( \text{div}(F) \) the divergence of \( F \). It is in two dimensions, where \( F(x,y) = (f(x,y), g(x,y)) \) given by the formula \( \text{div}(F)(x,y) = f_x(x,y) + g_y(x,y) \).

\[ \text{A differential equation } \frac{dx}{dt} = F(x) \text{ is area-preserving if and only if } \text{div}(F)(x,y) = 0 \text{ for all points in the plane.} \]

PROOF. By the change of variable formula \( \int_{T_t(\Omega)} dA = \int_{\Omega} |\text{det}(DT(x))| dA \), where \( DT \) is the Jacobean matrix of the transformation \( T \) at \( x \). Therefore \( \frac{d}{dt} \int_{T_t(\Omega)} dA = \int_{\Omega} \frac{d}{dt} |\text{det}(DT(x))| dA = \int_{\Omega} \text{div}(F(x)) dA \). (We could get rid of the absolute value because 1 + \text{div}(F) \) is positive for small \( t \).

2. PROOF. Define \( G(x,y,t) = (f(x,y), g(x,y)) \) and a tube like region \( \{(x(t), y(t), t) : x(0),y(0) \in \Omega, 0 < t < \tau \} \) in space-time. Applying the divergence theorem using \( \text{div}(G) = \text{div}(F) \) we get that the flux through the cylindrical walls is zero and the flux through the bottom is \( -\text{area}(G) \). The flux through the top is \( \text{area}(F(A)) \) and \( \text{area}(A) = \int_{T_t(\Omega)} \text{div}(F(x,y)) dA \).

This elegant proof does not need the coordinate change formula.

DISSIPATIVE SYSTEMS. If \( \text{div}(F) < 0 \) in a region, then area is shrinking. You will explore some of the consequences of dissipation in the homework. Here just an example:

**PROPOSITION.** In a region with \( \text{div}(F) < 0 \), there are no sources or elliptic equilibrium points.

**PROOF.** If \( (x_0, y_0) \) is the equilibrium point, then \( \text{div}(F) = \lambda_1 + \lambda_2 \). At sources, the real part of both \( \lambda_1 \) and \( \lambda_2 \) are positive. At elliptic equilibrium points, \( \lambda_1 \) and \( \lambda_2 \) are purely imaginary and the sum is 0.

EQUILIBRIUM POINTS. Points, where \( F(x,y) = (0,0) \) are called equilibrium points. An equilibrium point is called hyperbolic, if no eigenvalue has a real part equal to 0. Bifurcations can happen, when an eigenvalue passes through the axes \( \text{Re}(\lambda) = 0 \). In the hyperbolic case, one can conjugate the system near the equilibrium point to a linear system. This is a continuous version of the Sternberg-Grobman-Hartman theorem.

NULLCLINES. In two dimensions, we can draw the vector field by hand: attaching a vector \( f(x,y), g(x,y) \) at each point \((x,y)\). To find the equilibrium points, it helps to draw the nullclines \( f(x,y) = 0 \), \( g(x,y) = 0 \). The equilibrium points are located on intersections of nullclines. The eigenvalues of the Jacobians at equilibrium points allow to draw the vector field near equilibrium points. This information is sometimes enough to draw the vector field by hand.

**EXAMPLE: COMPETING SPECIES.** The system \( \dot{x} = (6 - 2x - y), \dot{y} = y(4 - x - y) \) has the nullclines \( x = 0, y = 0, 2x + y = 6, x + y = 5 \). There are 4 equilibrium points \((0,0), (3,0), (0,4), (2,2) \). The Jacobian matrix of the system at the point \((x_0, y_0)\) is
\[ \begin{bmatrix} 6 - 4x_0 - y_0 & -y_0 \\ -y_0 & 4 - x_0 - 2y_0 \end{bmatrix} \]
Without interaction, the two systems would be logistic systems \( \dot{x} = x(6 - 2x), \dot{y} = y(4 - y) \). The additional \(-xy\) part is due to the competition. If both \( x \) and \( y \) become large, then this produce resource problems for both species.

<table>
<thead>
<tr>
<th>Equilibrium</th>
<th>Jacobian</th>
<th>Eigenvalues</th>
<th>Nature of equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td>6 0</td>
<td>( \lambda_1 = 6, \lambda_2 = 4 )</td>
<td>Unstable source</td>
</tr>
<tr>
<td>(3,0)</td>
<td>-6 -3</td>
<td>( \lambda_1 = -6, \lambda_2 = 1 )</td>
<td>Hyperbolic saddle</td>
</tr>
<tr>
<td>(0,4)</td>
<td>2 0</td>
<td>( \lambda_1 = 2, \lambda_2 = -4 )</td>
<td>Hyperbolic saddle</td>
</tr>
<tr>
<td>(2,2)</td>
<td>-4 -2</td>
<td>( \lambda_1 = -3 \pm \sqrt{5} )</td>
<td>Stable sink</td>
</tr>
</tbody>
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