THE POINCARE BENDIXON THEOREM

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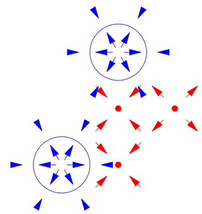
ABSTRACT. The Poincaré-Bendixon theorem tells that the fate of any bounded solution of a differential equation in the is to convergence either to an attractive fixed point or to a limit cycle. This theorem **rules out "chaos" for differential equations in the plane**.

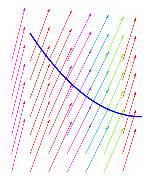
THEOREM (Poincare-Bendixon). Given a differential equation $\frac{d}{dt}x = F(x)$ in the plane. Assume $x(t)$ is an solution curve which stays in a bounded region. Then either $x(t)$ converges for $t \to \infty$ to an equilibrium point where $F(x) = 0$, or it converges to a single periodic cycle.



PRELIMINARIES.

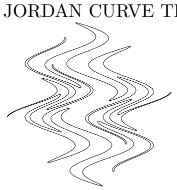
CYCLES, EQUILIBRIA AND CYCLES. Points x, where $F(x) = 0$ are called **equilibrium points** for the differential equation $\frac{dx}{dt}x = F(x)$. If a solution starts at an equilibrium point, it stays at the equilibrium point for ever. If $x(t)$ is a solution curve and $x(t+T) = x(t)$ for some $T > 0$, then the curve is called a **cycle**. Note that we do not include equilibrium points in this definition. The minimal time T for which $x(t+T) = x(T)$ is called the **period** of the cycle.

TRANSVERSE CURVES. A smooth curve $\gamma(s) \in R^2$ is called **transverse** to the vector field $x \mapsto F(x)$ if at every point $x \in \gamma$, the vector $F(x)$ and at least one tangent vector of γ passing through x are linearly independent.

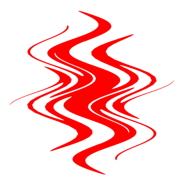


OMEGA LIMIT SET. The **omega limit set** $\omega^+(x_0)$ of an orbit $x(t)$ passing through x_0 is the set of points x, for which there exists a sequence of times t_n such that $x(t_n)$ converges to x. Equivalent is the mathematical statement $\omega^+(x_0) = \bigcap_{s \geq 0} \overline{\{x(t) \mid t \geq s\}}$, where $\overline{A}$ is the **closure** of a set A. If the ω-limit set of an orbit is a cycle, it is called a **limit cycle**.

JORDAN CURVE THEOREM.



A **Jordan curve** is a simple closed curve in the plane. "Simple" means that the curve should not have selfintersections or be tangent to itself at any point. The **Jordan curve theorem** assures that such a curve devides the plane into two disjoint regions, the "inside" and the "outside". This seemingly elmentary fact is surprisingly hard to prove.



EXAMPLE OF LIMIT CYCLE. The differential equation given in polar coordinates as

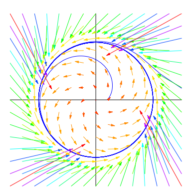
$$\frac{dr}{dt} = r(1-r^2), \frac{d\theta}{dt} = 1$$

is with $x = r\cos(\theta),\ y = r\sin(\theta)$ equivalent to

$$\frac{dx}{dt} = \frac{dr}{dt}\cos(\theta) - r\sin(\theta)\frac{d\theta}{dt} = (1-(x^2+y^2))x - y$$

$$\frac{dy}{dt} = \frac{dr}{dt}\sin(\theta) + r\cos(\theta)\frac{d\theta}{dt} = (1-(x^2+y^2))y + x$$

In this example, all initial conditions away from the origin will converge to the limit cycle.



EXAMPLE OF ATTRACTIVE POINT. The differential equation given in polar coordinates as

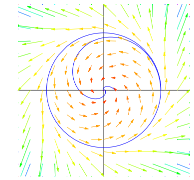
$$\frac{dr}{dt} = r(r^2-1), \frac{d\theta}{dt} = 1$$

is with $x = r\cos(\theta),\ y = r\sin(\theta)$ equivalent to

$$\frac{dx}{dt} = \frac{dr}{dt}\cos(\theta) - r\sin(\theta)\frac{d\theta}{dt} = ((x^2+y^2)-1)x - y$$

$$\frac{dy}{dt} = \frac{dr}{dt}\sin(\theta) + r\cos(\theta)\frac{d\theta}{dt} = ((x^2+y^2)-1)y + x$$

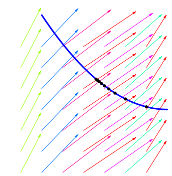
In this example, all initial conditions away from the limit cycle will converge to the origin or to infinity.



PROOF OF THE POINCARE-BENDIXON THEOREM. The aim is to show that if the omega limit set $\omega^+(x_0)$ is nonempty, then it either an equilibrium point or a closed periodic orbit.

(i) There are no equilibrium points on a transverse curve. The vector field f can therefore not reverse direction along the curve.

(ii) Let γ be a transverse curve. If a solution $x(t)$ crosses γ more than once, the successive crossing points form a monotonic sequence on the arc γ.
Proof. Denote by $x(t_1) = \gamma(s_1), x(t_2) = \gamma(s_2)$, the first two crossing times. We can assume that $s_2 \geq s_1$ because if this does not hold, one can reparametrize γ by $s' = 1 - s$ if $s_1 < s_2$. The union of the two smooth arcs $\{x(t) \mid t_1 \leq t \leq t_2\}$ and $\{\gamma(s) \mid s_1 \leq s \leq s_2\}$ is a closed piecewise smooth curve. By Jordan's curve theorem, such a curve divides the plane into two different regions. For $t > t_2$, the solution $x(t)$ stays in one of these regions. For the next crossing $x(t_3) = \gamma(s_3)$ one has therefore $s_3 \geq s_2$.

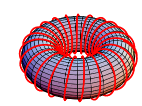


(iii) It follows from (ii) that no more than one point of any transverse arc γ can belong to the ω limit set $\omega^+(x_0)$.

(iv) Given $y_0 \in \omega^+(x_0)$. Because a solution $y(t)$ with $y(0) = y_0$ stays by assumption in a bounded region, the solution $y(t)$ is by the existence theorem for differential equations defined for all times. It stays in $\omega^+(x_0)$ because this set is invariant under the flow. Assume, there exists no stationary point in $\omega^+(x_0)$. There exists then a transverse arc γ passing through y_0. Because $\omega^+(x_0) \cap \gamma$ can have only one intersection and $y(t)$ returns arbitrary close to y_0, the orbit $\{y(t)\}$ through y_0 is a single periodic orbit.

DIFFERENT SURFACES. Does an anlogue of Poincare Bendixon hold also on other two dimensional spaces? The answer depends on the space. On the sphere, the answer is yes, on the torus, there are solutions which are neither asymptotic to a limit cycle or equilibrium point. An example of such a curve is $(t, \alpha t)$ mod 1 which is a solution of the differential equation

$$\frac{d}{dt}x = 1, \ \frac{d}{dt}y = \alpha \ .$$



Differential equations of the form

$$\frac{d}{dt}x = F(x,y), \ \frac{d}{dt}y = \alpha F(x,y) \ .$$

can even show some weak type of mixing. You explore the question a bit in a homework problem.