STRICT ERGODICITY (* not treated in class)  Math118, O. Knill

ABSTRACT. The irrational rotation on the circle is a minimal uniquely ergodic system. Other systems occurring in number theory have the same property.

ERGODICITY. A map $T$ is ergodic if for every function $f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}$ with finite $\sum_n c_n^2$, the condition $f(T) = f$ implies $f = \text{const}$.

THEOREM. For irrational $\alpha$, the map $T(x) = x + \alpha$ is ergodic.

PROOF. Comparing Fourier coefficients of $f(T)$ and $f$ gives $c_{n\alpha} = c_n$ so that $c_n = 0$ unless $n = 0$.

UNIQUE ERGODICITY. A continuous transformation $T$ on a compact topological space is uniquely ergodic if there is only one invariant measure $\mu$ of $T$.

KRONECKER-WEYL THEOREM. The only measure which is invariant under an irrational rotation is the length measure $dx$.

PROOF. A measure $\mu$ is a linear map from the space of all continuous functions $C(X)$ to $R$ given by $\mu(f) = \int f(x) \, d\mu(x)$. If $\mu$ is $T$ invariant, then $\mu(f(T)) = \mu(f)$ and by linearity $\mu\left(\frac{1}{n} \sum_{k=1}^n f(T^k)\right) = \mu(f)$. Because for $f = e^{inx}$ we have

$$\frac{1}{n} \sum_{k=1}^n f(T^k) = \frac{1}{n} \sum_{k=1}^n e^{in(x+kn)} = e^{inx} \frac{1 - e^{in\alpha}}{1 - e^{in\alpha}} = 0$$

also for any $f = \sum_n e^{inx}$ we have $\mu(f) = \mu\left(\frac{1}{n} \sum_{k=1}^n f(T^k)\right) \to c_0$ for $n \to \infty$ which implies $\mu(f) = c = f$. $f(x) \, dx$.

MINIMALITY. A map $T$ is called minimal, if every orbit of $T$ is dense.

THEOREM. The irrational rotation on the circle is minimal.

PROOF. This follows in a constructive way from Chebychevs theorem. For every $x$ and $y$ and $\varepsilon > 0$, there exists $n$ such that $|x + n\alpha - y| < \varepsilon$.

STRICT ERGODICITY. A map is called strictly ergodic, if it is both minimal and uniquely ergodic.

COROLLARY. The irrational rotation on the circle is strictly ergodic.

HIGHER DIMENSIONAL GENERALIZATION. The above statements go through word by word for a rotation $T(x) = x + \alpha$ with vectors $\alpha = (\alpha_1, \ldots, \alpha_d)$ for which $n \cdot \alpha = n_1 \alpha_1 + \cdots + n_d \alpha_d = 0$ implies $n = 0$. We call such vectors irrational. Functions of several variables have a Fourier expansion too: $f(x) = \sum_{n \in \mathbb{Z}^d} c_n e^{i\alpha_n x}$, where $n = (n_1, \ldots, n_d)$ runs over all lattice points in $\mathbb{Z}^d$.

COROLLARY. The irrational translation on the torus $T^d = R^d/\mathbb{Z}^d$ is strictly ergodic.

PROOF. We have shown both minimality as well as unique ergodicity.

THEOREM (FURSTENBERG) If $\alpha$ is irrational and $b_{ij} \in \mathbb{Z}, 1 \leq j < i \leq d$ real with $b_{i,i-1} \neq 0$. Then $T(x_1, \ldots, x_d) = (x_1 + \alpha_1 x_2 + b_{21} x_1, \ldots, x_d + b_{d1} x_1 + \cdots + b_{d,d-1} x_{d-1})$ defines a uniquely ergodic system on $T^d$.

It can be written as $\vec{x} \mapsto A \vec{x} + \vec{c}$, where $A = \begin{bmatrix} 1 & 0 & 0 \\ b_{12} & 1 & 0 \\ \vdots & \vdots & \ddots \\ b_{d1} & b_{d2} & \cdots & 1 \end{bmatrix}$ and $\vec{c} = (c_1, \ldots, c_{d-1})$.

PROOF. Fourier theory shows that $T$ is ergodic: $f(T) = \sum_n c_n e^{i\alpha x + n\alpha x} \to (n \cdot \alpha = (n_1, \ldots, n_d)) \alpha \cdot x$. Comparing Fourier coefficients gives $c_{n\alpha} = c_n e^{2\pi in\alpha x}$ which implies $\alpha = (n_1, \ldots, n_d) = 0$ and therefore $c_n = c_n e^{2\pi in\alpha x}$ which implies that $c_n = 0$ unless $n = 0$.

Uniqueness is shown with induction to $d$. We know it for $d = 1$, where the system is an irrational rotation. To prove the result in dimension 2, write $T(\vec{x}, x_d) = (S(\vec{x}), x_d + A \cdot \vec{x})$. Note that $S$ does not depend on $x_d$. By induction, $S$ is uniquely ergodic on $T^{d-1}$. Given invariant measure $\mu$ for $T$, the projection of $\mu$ on $T^{d-1}$ is invariant, and by induction assumption the volume measure $dx_1, \ldots, dx_{d-1}$.

Because $T$ commutes with $R(\vec{x}, y) = (\vec{x}, y + \beta)$, a $T$ invariant measure must also be $R_\beta$ invariant for every $\beta$. By Birkhoffs ergodic theorem, we know that $\mu$ almost all points $x = (x, x_d)$ are generic in the sense that $\mu(\text{fn}(x)) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n f(T^k x)$. Assume $x = (x, y)$ is generic. Then also $(x,y + \beta)$ is generic.

A uniquely ergodic system on the torus which preserves the volume measure $dx_1, \ldots, dx_d$ is automatically minimal: if there were an orbit which were not dense, then its closure $Y$ would be a $T$ invariant set which is not the entire torus. This set would carry an other invariant measure.

ILLUSTRATION. Let's see this in the case $T(x, y) \to (x + \alpha, y + \beta) \to (x + \alpha, y + \beta)$ where $(x, y)$ is an orbit, then the distribution of $x_n$ on the first coordinate are the measure $dx$. Assume two different points $(x_1, y_1), (x_2, y_2)$ with irrational $\beta$ produce measures $\mu(\vec{x}, \beta)$, $\nu(\vec{x}, \gamma)$ which must coincide.

APPLICATION: Let $p(x)$ be polynomial of degree $n$. Define $p_0(x) = p(x), p_{n-1} = p_n(x) = p_n - p_{n-1}(x) - \cdots - p_0(x)$. Each $p_i$ is a polynomial of degree $i$. With

$$T_p = \begin{bmatrix} x_1 + \alpha & x_2 + \alpha & \cdots & x_n + \alpha \\ x_2 + \alpha & x_3 + \alpha & \cdots & x_n + x_1 + \alpha \\ \vdots & \vdots & \ddots & \vdots \\ x_n + \alpha & x_1 + \alpha & \cdots & x_2 + \alpha \end{bmatrix}$$

we have $T_p(p_0(n), p_2(n), \ldots, p_{n-1}(n)) = (p_0(n+1), \ldots, p_0(n+1))$.

COROLLARY. If $p = a_0 x^n + \cdots + a_n x + a_0$ is a polynomial of degree $d$ and assume $a_0$ is irrational, then $T_a$ is a uniquely ergodic transformation on the $n$ dimensional torus which preserves volume $dx_1, \ldots, dx_n$.

QUESTION. Are polynomials the only functions $f$ for which one can describe $f(n)$ mod 1 by a finite dimension system?

EXAMPLES.
1) For $f(x) = \sqrt{2}$. What dynamical system does $\sqrt{2} \mod 1$ generate?
2) Does $f(x) = \exp(x)$ generate an infinite dimensional system?
3) If $f(x)$ is a $k$-periodic function, then $f(n)$ is periodic too For $f(x) = \sin(2\pi x\alpha)$ with irrational $\alpha$, then $f(n)$ is an almost periodic sequence. The system on the torus $(x, y) \to (x + \alpha, \sin(2\pi x))$ allows to read of $f(n)$ in one coordinate.

4) For rational functions like $f(x) = x/(1 + x^2)$, the system has a fixed point which attracts all points.

OTHER STRICTLY ERGODIC SYSTEMS. Any factor of a strictly ergodic system is strictly ergodic. This applies to symbolic dynamics.

Doing symbolic dynamics with a strictly ergodic system produces strictly ergodic subshifts. Let $A_1, A_2, \ldots, A_d$ be a partition of $T^d$ into subsets, define $S(x)_n = k$ if $T^n(x) \in A_k$. This defines a subshift which is strictly ergodic.

EXAMPLES. Sturmian sequences $x_n = \frac{1}{A}(x + n\alpha)$ and especially the Fibonacci sequence are uniquely ergodic subshifts. Applying cellular automata maps on such subshifts generates new subshifts which are strictly ergodic. CA maps preserve both minimality as well as unique ergodicity.