

THE MANDELBROT SET

Math118, O. Knill

ABSTRACT. This is a proof a theorem of Douady and Hubbard assuring that the Mandelbrot set is connected. The proof needs some concepts from topology and complex analysis and topology.

BÖTTCHER-FATOU LEMMA.

Assume $f(z) = z^k + a_{k+1}z^{k+1} + \dots$ with $k \geq 2$ is analytic near 0. Define $\phi_n(z) = (f^n(z))^{1/k^n} = z + a_1z^2 + \dots$. In a neighborhood U of $z = 0$ $\phi = \lim_{n \rightarrow \infty} \phi_n(z) : U \rightarrow B_r(0)$ satisfies $\phi \circ f \circ \phi^{-1}(z) = z^k$ and $\phi(0) = 0$ and $\phi'(0) = 1$.

PROOF. We show that ϕ_n converges uniformly. The properties $\phi(f(z)) = \phi(z)^k$ as well as $\phi(0) = 0$ and $\phi'(0) = 1$ follow from the assumptions. The function

$$h(z) := \log\left(\frac{f(z)^{1/k}}{z}\right)$$

with the chosen root $f(z)^{1/k} = z + O(z^2)$ is analytic in a neighborhood U of 0 and there exists a constant C such that $|h(z)| \leq C|z|$ for $z \in U$. U can be chosen so small that $f(U) \subset U$ and $|f(z)| \leq |z|$. We can write $\phi(z)$ as an infinite product

$$\phi(z) = z \cdot \frac{\phi_1(z)}{z} \cdot \frac{\phi_2(z)}{\phi_1(z)} \cdot \frac{\phi_3(z)}{\phi_2(z)} \dots$$

This product converges, because $\sum_{n=0}^{\infty} \log \frac{\phi_{n+1}(z)}{\phi_n(z)}$ converges absolutely and uniformly for $z \in U$:

$$\left| \log \frac{\phi_{n+1}(z)}{\phi_n(z)} \right| = \left| \log \left[\frac{(f \circ f^n(z))^{1/k^{n+1}}}{f^n(z)} \right] \right| = \frac{1}{k^{n+1}} \cdot |h(f^n(z))| \leq \frac{1}{k^{n+1}} C \cdot |f^n(z)| \leq \frac{C \cdot |z|}{k^{n+1}}$$

COROLLARY (*). If $c \mapsto f_c(z)$ is a family of analytic maps such that $c \mapsto f_c(z)$ is analytic for fixed z , and c is in a compact subset of C , then the map $(c, z) \mapsto \phi_c(z)$ is analytic in two variables.

PROOF. Use the same estimates as in the previous proof: the maps $(c, z) \mapsto \phi_n(c, z)$ are analytic and the infinite product converges absolutely and uniformly on a neighborhood U of 0.

PROPOSITION The Julia set J_c is a compact nonempty set.

PROOF.

(i) The Julia set is bounded: the Lemma of Boettcher-Fatou implies that every point z with large enough $|z|$ converges to ∞ . This means that a whole neighborhood U of z escapes to ∞ . In other words, the family $\mathcal{F} = \{f_c^n\}_{n \in \mathbb{N}}$ is normal, because every sequence in \mathcal{F} converges to the constant function ∞ .

(ii) The Julia set is closed: this follows from the definition, because the Fatou set F_c is open.

(iii) Assume the Julia set were empty. The family $\mathcal{F} = \{f_c^n\}$ would be normal on \overline{C} . This means that for any sequence f_n in \mathcal{F} , there is a subsequence f_{n_k} converging to an analytic function $f : \overline{C} \rightarrow \overline{C}$. Because such a function can have only finitely many zeros and poles, it must be a rational function P/Q , where P, Q are polynomials. If $f_{n_k} \rightarrow f$, there are eventually the same number of zeros of f_{n_k} and f . But the number of zeros of f_{n_k} (counted with multiplicity) grows monotonically. This contradiction makes $J_c = \emptyset$ impossible.

COROLLARY. The Julia set J_c is contained in the **filled in Julia** set K_c , the union of J_c and the bounded components of the Fatou set F_c .

PROOF. Because J_c is bounded and f -invariant, every orbit starting in J_c is bounded and belongs by definition to the filled-in Julia set. If a point is in a bounded component of F_c , its forward orbit stays bounded and it belongs to the filled in Julia set. On the other hand, if a point is not in the Julia set or a bounded component of F_c , then it belongs to an unbounded component of the Fatou set F_c .

GREEN FUNCTION. A continuous function $G : C \mapsto R$ is called the potential theoretical **Green function** of a compact set $K \subset C$, if G is **harmonic** outside K , vanishing on K and has the property that $G(z) - \log(z)$ is bounded near $z = \infty$.

The Green function G_c exists for the filled-in Julia set K_c of the polynomial f_c . The map $(z, c) \mapsto G_c(z)$ is continuous.

PROOF. The Boettcher-Fatou lemma assures the existence of the function ϕ_c conjugating f_c with $z \mapsto z^2$ in a neighborhood U_c of ∞ . Define for $z \in U_c$

$$G_c(z) = \log |\phi_c(z)|.$$

This function is harmonic in U_c and growing like $\log |z|$ because by Boettcher satisfies $|f_c^n(z)| \geq C|z|^{2^n}$ for some constant C and so

$$G_c(z) = \lim_{n \rightarrow \infty} \frac{1}{2^n} \log |f_c^n(z)|.$$

Although G_c is only defined in U_c , there is one and only one extension to all of C which is continuous and satisfies

$$G_c(z) = G_c(f_c(z))/2. \tag{1}$$

In fact, we define $G_c(z) = 0$ for $z \in K_c$, and $G_c(z) = G(f_c^n(z))/2^n$ otherwise, where n is large enough so that $f_c^n(z) \in U$. We know from this extension that G_c is a smooth real analytic function outside K_c . From the **maximum principle**, we know that $G_c(z) > 0$ for $z \in C \setminus K_c$. We have still to show that G_c is continuous in order to see that it is the Green function. The continuity follows from the stronger statement:

$(z, c) \mapsto G_c(z)$ is jointly continuous. $G_c^{-1}([0, \epsilon])$ is open in C^2 for all $\epsilon > 0$ if and only if there exists n such that

$$A_n := \{(c, z) \mid G_c(f_c^n(z)) \geq 2^n \epsilon\}$$

is closed $\forall \epsilon > 0$. Given $r > 0$. There exists a ball of radius b which contains all the sets K_c for $|c| \leq r$. For $R \geq G_r(b)$, all the solutions ξ of $G_c(\xi) \geq R$ satisfy $|\xi| \geq b$ if $|c| \leq r$. The set $B = \{(c, \xi) \mid G_c(\xi) \geq R\} \cap \{|c| \leq r\}$ is closed. For n large enough, also $A_n \cap \{|c| \leq r\}$ is closed and A_n is closed.

THEOREM (DOUADY-HUBBARD). The Mandelbrot set M is connected.

CORE OF THE PROOF. The Böttcher function $\phi_c(z)$ can be extended to

$$S_c := \{z \mid G_c(z) > G_c(0)\}.$$

Continue defining $\phi_c(z) := \sqrt{\phi_c(z^2 + c)}$ to get ϕ_c having defined in larger and larger regions. This can be done as long as the region $\phi_c^{-1}(\{r\})$ is connected (this assures that the derivative of ϕ_c is not vanishing). Because Equation (1) gives $G_c(c) = 2G_c(0) > G_c(0)$, every c is contained in the set S_c and the map

$$\Phi : c \mapsto G_c(c)$$

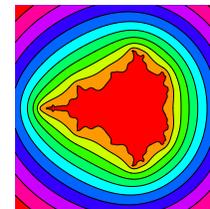
is well defined. It is analytic outside M and can be written as

$$\Phi(z) = \lim_{n \rightarrow \infty} [f_c^n(c)]^{1/2^n}.$$

Claim:

$$\Phi : \overline{C} \setminus M \rightarrow \overline{C} \setminus \overline{D}$$

is an analytic diffeomorphism, where $\overline{C} = C \cup \{\infty\}$ is the Riemann sphere. (This implies that the complement of M is simply connected in \overline{C} , which is equivalent to the fact that M is connected). The picture to the right shows the level curves of the function $\phi_6(c) = [f_6^6(c)]^{1/64}$. The function $\phi_6(z)$ is already close to the map $\Phi(z)$ in the sense that the level sets give a hint about the shape of the Mandelbrot set.



(1) Φ is analytic outside M . This follows from the Corollary.

(2) For $c_n \rightarrow M$, we have $|\Phi(c_n)| \rightarrow 1$. Proof. Continuity of the Green function.

(3) The map Φ is proper. (A map is called **proper** if the inverse of any compact set is compact). Given a compact set $K \subset C \setminus D$. The two compact sets D and K have positive distance. Assume $\phi^{-1}(K)$ is not compact. Then, there exists a sequence $c_n \in \Phi^{-1}(K)$ with $c_n \rightarrow c_0 \in M$ so that $|\Phi(c_n)| \rightarrow 1$. This is not possible because $\Phi(c_n) \in K$ is bounded away from D .

(4) The map Φ is open (it maps open sets into open sets). This follows from the fact that Φ is analytic. (This fact is called **open mapping theorem** (see Conway p. 95))

(5) The map Φ maps closed sets into closed sets. A proper, continuous map $\Phi : X \rightarrow Y$ between two locally compact metric spaces X, Y has this property. Proof. Given a closed set $A \subset X$. Take a sequence $\Phi(a_n)$ in $\Phi(A)$ which converges to $b \in Y$. Take a compact neighborhood K of b (use local compactness of Y). Then $\Phi^{-1}(K \cap \Phi(A))$ is compact and contains almost all a_n . The sequence a_n contains therefore an accumulation point $a \in X$. The continuity implies $\Phi(a_n) \rightarrow \Phi(a) = b$ for a subsequence so that $b \in \Phi(K)$. Consequently $\Phi(K)$ is closed.

(6) Φ is surjective. The image of $\Phi(\overline{C} \setminus M)$ is an open subset of set $\overline{C} \setminus \overline{D}$ because Φ is open. The image of the boundary of M is (use (5)) a closed subset of $\overline{C} \setminus D$ which coincides with the boundary of D because the boxed statement about the the Green function showed $G_c(c) \rightarrow 0$ as $c \rightarrow M$.

(7) Φ is injective. Because the map Φ is proper, the inverse image $\phi^{-1}(s)$ of a point s is finite. There exists therefore a curve Γ enclosing all points of $\Phi^{-1}(s)$. Let $\sharp A$ denote the number of elements in A . By the **argument principle** (see Alfors p. 152), we have

$$\sharp(\phi^{-1}(s)) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Phi'(z)}{\Phi(z) - s} dz$$

and this number is locally constant. Given $M > 0$, we can find a curve Γ which works simultaneously for all $|s| \leq M$. Because Φ is surjective and $\sharp(\phi^{-1}(\infty)) = 1$, we get that $\sharp(\phi^{-1}(s)) = 1$ for all $z \in C \setminus \overline{D}$ and ϕ is injective.

(8) The map Φ^{-1} exists on $C \setminus D$ and is analytic. Because an injective, differentiable and open map has a differentiable inverse, (this is called **Goursat's theorem**), the inverse is analytic.

NOTATIONS.

- $f(z)$ is **analytic** in a set U if the derivative $f'(z) = \lim_{w \rightarrow 0} (f(z+w) - f(z))/w$ of f exists at every point in U . This means that for $f(z) = f(x+iy) = u(x+iy) + iv(x+iy)$ the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are all continuous real-valued functions on U . In that case $u(x,y), v(x,y)$ are **harmonic**: $u_{xx} + u_{yy} = 0$.
- $B_r(z) = \{w \mid |z-w| < r\}$ is a neighborhood of z called an **open ball**.
- A sequence of analytic maps f_n **converges uniformly** to f on a compact set $K \subset U$, if $f_n \rightarrow f$ in $C(K)$, which means $\max_{x \in K} |f_n(x) - f(x)| \rightarrow 0$.
- A family of analytic maps \mathcal{F} on U is called **normal**, if every sequence $f_n \in \mathcal{F}$ has a subsequence which converges uniformly on any compact subset of U . The limit function f does not need to be in \mathcal{F} . With respect to the topology of convergence on compact subsets normality is precompactness in this topology: \mathcal{F} is normal, if and only if its closure is compact. The **theorem of Arzela-Ascoli** (see Alfors p. 224) states says that normality of \mathcal{F} is equivalent to the requirement that each f is equicontinuous on every compact set $K \subset U$ and if for every $z \in U$, the set $\{f(z) \mid f \in \mathcal{F}\}$ is bounded. z is part of the **Fatou set** of f , $\{f^n\}_{n \in \mathbb{N}}$ is normal in some neighborhood of z . The **Julia set** is the complement of the Fatou set.
- A set is called **locally compact**, if every point has a compact neighborhood. In the plane, a set is compact if and only if it is bounded and closed. A subset is closed, if and only if its complement is open. A subset U is open, if for every point x in U there is a ball $B_r(x)$ which still belongs to U .

SOME HISTORY:

In 1879, **Arthur Cayley** poses the problem to study the regions in the plane, where the Newton iteration converges to some root.



Gaston Julia (1893-1978) and **Pierre Fatou** (1879-1929) both worked already 90 years ago on the iteration of analytic maps. Julia and Fatou sets are called after them. Julia and Fatou were both competed for the 1918 'grand prix' of the academie of sciences and produced similar results. This produced a priority dispute. Julia lost his nose in world war I and had since to wear a leather strap across his face. He had continued with his research in the hospital.



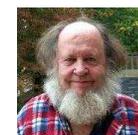
Robert Brooks and **Peter Matelski** produce in 1978 the first picture of the Mandelbrot set in the context of Kleinian groups. Their paper had the title "The dynamics of 2-generator subgroups of $PSL(2, C)$ ". The defined $\tilde{M} = \{c \mid f_c \text{ has a stable periodic orbit}\}$. This set is now called **Brooks-Matelski set** and is now believed to be the interior of the Mandelbrot set M . If the later were locally connected, this would be true: $\text{int}(M) = \tilde{M}$.



John Hubbard made better pictures of a quite different parameter space arising from Newton's method for cubics. Hubbard was inspired by a question from a calculus student. **Benoit Mandelbrot**, perhaps inspired by Hubbard, made corresponding pictures in 1980 for quadratic polynomials. He conjectured the set M is disconnected because his computer pictures showed "dust" with no connections to the main body of M . It is amusing that the journals editorial staff removed that dust, assuming it was a problem of the printer. John Milnor writes in his book of 1991: "Although Mandelbrot's statements in this first paper were not completely right, he deserves a great deal of credit for being the first to point out the extremely complicated geometry associated with the parameter space for quadratic maps. His major achievement has been to demonstrate to a very wide audience that such complicated fractal objects play an important role in a number of mathematical sciences."



Adrien Douady and **John Hubbard** prove the connectivity of M in 1982. This was a mathematical breakthrough. In that paper the name "Mandelbrot set" was introduced. The paper provided a firm foundation for its mathematical study. We followed on this handout their proof. Note that the Mandelbrot set is also **simply connected**, but this is easier to show. Both statements use that a subset of the plane is connected if and only if the complement is simply connected.



Evenso one of the first things which comes in mind, when talking about fractals is the Mandelbrot set. It is not a "fractal": in 1998, **Mitsuhiro Shishikura** has shown that its Hausdorff dimension of M is 2. (M. Shishikura, "The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets, Annals of Mathematics 147 (1998), 225-267.)



Also for higher dimensional polynomials, one can define Julia and Mandelbrot sets. For cubic polynomials $f_{a,b}(z) = z^3 - 3a^2z + b$, define the **cubic locus set** $\{(a,b) \in C^2 \mid K_{a,b} \text{ is connected}\}$, where $K_{a,b}$ is the **prisoner set** $K_{a,b} = \{z \mid f_{a,b}^n(z) \text{ stays bounded}\}$. **Bodil Branner** showed around 1985, that the cubic locus set is connected. This generalizes the main result discussed in this handout.



OPEN PROBLEMS. The major open problem is whether the Mandelbrot set is locally connected or not. A subset M of the plane is called **locally connected**, if at every point $x \in M$ if every neighborhood of x contains a neighborhood, in which M is connected. A locally connected set does not need to be connected (two disjoint disks in the plane are locally connected but not connected). A connected set does not need to be locally connected. An example is the union of the graph of $\sin(1/x)$ and the y -axes.