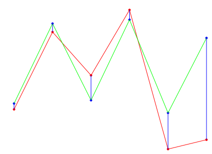
2/9/2005: LYAPUNOV EXPONENT — Math118, O. Knill

ABSTRACT. We demonstrate that the **logistic map** $f(x) = 4x(1-x)$ is chaotic in the sense that the Lyapunov exponent, a measure for sensitive dependence on initial conditions is positive.

LYAPUNOV EXPONENT. For an orbit of f with starting point x, we define the **Lyapunov exponent** as

$$\lambda(f,x) = \lim_{n\to\infty} \frac{1}{n} \log|(f^n)'(x)|$$

where $(f^n)'$ is the derivative of the n'th iterate (f^n). Remark. It turns out that usually, the limit exists. If not, one should replace lim with lim inf, the smallest accumulation point of the sequence. Choosing lim inf instead of lim sup has nicer analytic properties.



A BETTER FORMULA. The function $f^n(x)$ becomes complicated already for small n. The following formula is more convenient to compute the Lyapunov exponent of an orbit through x_0:

$$(f^n)'(x) = f'(x_{n-1})...f'(x_1)f'(x_0)$$

PROOF. Use induction: for $n = 1$, the claim is obvious. If we differentiate $f^n(x_0) = f^{n-1}(f(x_0))$ we get $(f^{n-1})'(x_1)f'(x_0)$, then use the induction assumption $(f^{n-1})'(x_1) = f'(x_{n-1})...f'(x_1)$. Therefore:

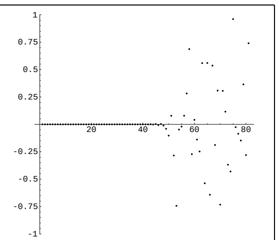
$$\lambda(f,x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log|f'(x_k)|$$

EXAMPLE. For the logistic map, we compute the Lyapunov exponent by taking a large n and form

$$\frac{1}{n}\left[\log|c(1-2x_{n-1})| + \log|c(1-2x_{n-2})| + \cdots + \log|c(1-2x_0)|\right] .$$

WHAT DOES THE LYAPUNOV EXPONENT MEASURE? If x and y are close, then $|f(y)-f(y)| \sim |f'(x)||x-y|$, if x and y are close. because Taylors formula assures $f(y) = f(x) + f'(x)(y-x)$ plus something of the order $(y-x)^2$. If x_n is the orbit of x and y_n is the orbit of y, then for a fixed n, we have $|x_n - y_n| \sim |(f^n)'(x)||x_0 - y_0|$ if x_0 and y_0 are close together.
The Lyapunov exponent is a quantitative number which indicates the **sensitive dependence on initial conditions**. It measures the exponential rate at which errors grow. If the Lyapunov exponent is $\log|c|$ then you can expect an error $c^n\epsilon$ after n iterations, if ϵ was the initial error.

EXAMPLE. We will see below that the Lyapunov exponent of $f(x) = 4x(1-x)$ is $\log|2|$. If your initial error is $\epsilon = 10^{-16}$, then we have after n iterations an error $2^n\epsilon$ which is of the order 1 for $n = 53$. To the right we see the difference $x_n - y_n$ between two orbits of the map $f = f_4$ which have an initial condition $|x_0 - y_0| = 10^{-16}$. You see that after about 50 iterations, the error has grown so much that it becomes visible.



LYAPUNOV EXPONENT OF PERIODIC ORBIT. If $x_0, x_1, ..., x_n = x_0$ is a periodic orbit of period n, then

$$\lambda(f,x) = \frac{1}{n}(f^n)'(x) = \frac{1}{n}\left(\log|f'(x_{n-1})| + \log|f'(x_{n-2})| + ... + \log|f'(x_0)|\right) .$$

PROOF. We we have to show that the sequence $s_k = \frac{1}{k}(\log|f'(x_0)| + \log|f'(x_1)| + ... + \log|f'(x_k)|)$ converges to the right hand side which is s_n. If k is a multiple of n, then $s_k = s_n$. If M is the maximum of all the numbers $\log|f'(x_i)|$, then $|s_j| \leq jM$ for $k = 1, ..., n$ and $s_k - \lambda(f,x) \leq (nM)/k$.

EXAMPLES.

- The Lyapunov exponent of the fixed point 0 is $\log(c)$. It is negative for $c < 1$ and positive for $c > 1$.
- The Lyapunov exponent of the fixed point $1 - 1/c$ of the logistic map f_c is $\log|f'(1-1/c)| = \log|2-c|$.

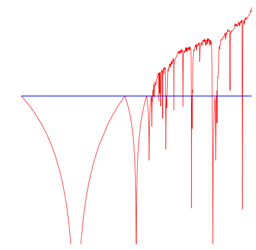
LYAPUNOV EXPONENT OF AN ATTRACTIVE PERIODIC ORBIT. The Lyapunov exponent of an attractive periodic orbit is negative.

PROOF. We have $\lambda(f,x) = \frac{1}{n}\log|(f^n)'(x)|$. We have seen that for an attractive periodic orbit, $|(f^n)'| < 1$.

It follows that the Lyapunov exponent of an orbit which is attracted to a periodic orbit is negative too.

LYAPUNOV EXPONENT AND BIFURCATION. A periodic point can only bifurcate if its Lyapunov exponent is zero.

LYAPUNOV EXPONENT OF THE LOGISTIC MAP.
The picture to the right shows the Lyapunov exponent of an orbit starting at x_0 in dependence of c. You see that this graph looks very complicated. If the Lyapunov exponent is negative, we typically have an attractive periodic orbit.



It is difficult to say something about the Lyapunov exponent of a specific parameter. We know what happens for $c = 4$ and we know what happens in case of an attractive periodic orbit. If an attractive periodic orbit exists, there is an entire interval, where the Lyapunov exponent is negative. It has only recently been shown that there is a dense set of parameters for which the Lyapunov exponent is negative. This means, we don't find a single interval in $[0,4]$ on which the Lyapunov exponent is positive.

CONJUGATION OF MAPS. Two interval maps T and S are **conjugated**, if there exists an invertible map U from the interval onto itself such that $T(U(x)) = U(S(x))$. If both maps are differentiable maps, one usually requires the map U to be smooth too.

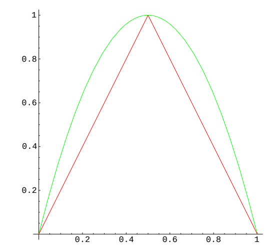
COROLLARY: The Lyapunov exponents of corresponding orbits of two conjugated interval maps are the same.

More precisely, if $\lambda(f,x)$ is the Lyapunov exponent of the orbit of f through x, and $\lambda(g,y)$ is the Lyapunov exponent of the orbit of g through $y = h(x)$, and $gh(x) = hf(x))$ is the conjugation, then $\lambda(f,x) = \lambda(g,y)$.

Proof: This is an application the chain rule.

ULAM AND TENT ARE CONJUGATED MAPS: the Ulam map $T(x) = 4x(1-x)$ is conjugated to the tent map $S(x) = 1 - 2|x - 1/2|$ with the conjugation $U(x) = \frac{1}{2} - \frac{1}{\pi}\arcsin(1-2x)$ and $U^{-1}(x) = \frac{1}{2} - \frac{1}{2}\cos(\pi x)$.

PROOF. To check that $UTU^{-1}(x) = S(x)$, we show $UT(x) = S(U(x))$. One can get rid of the absolute value by distinguishing the cases $x > 1/2$ and $x < 1/2$. We have $U(T(x)) = \frac{1}{2} - (\arcsin(1 - 8(1-x)x)))/\pi$ and $SU(x)) = 1 + 2(\arcsin(1-2x))/\pi$ for $x \in [1/2, 1]$. To verify the identity, we check that both sides are 1 for $x = 1/2$ and that $\frac{d}{dx}\arcsin(1 - 8x + 8x^2) = -\frac{d}{dx}2\arcsin(1-2x)$. The last identity is best checked by squaring both sides and using $\arcsin'(x) = 1/\sqrt{1-x^2}$. The identity on $[0, 1/2]$ is solved in the same way.



LYAPUNOV EXPONENT OF THE ULAM MAP.

THEOREM. For all but a countable set of initial conditions x_0, the Lyapunov exponent of $f(x) = 4x(1-x)$ with initial condition x_0 is equal to $\log(2)$.

The tent map $S(x) = 1 - 2|x - 1/2|$ is piecewise linear. The derivative $S'(x)$ is either 2 or -2. Since $\log|S'(x_k)| = \log(2)$, the map has the Lyapunov exponent $\log(2c)$ for orbits, which do not hit one of the discontinuities. Most initial points do not hit the discontinuity because there is only a countable set of initial conditions for which this can happen.

Because the map is conjugated to T_4, the Lyapunov exponent of f_4 is $\log(2)$ too by the corollary.