**STABLE AND UNSTABLE MANIFOLDS**

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**ABSTRACT.** Near a hyperbolic point, one can conjugate the map by its linearization. This conjugation defines local curves through the origin which are invariant. These stable and unstable manifolds intersect in general to points which are called homoclinic points. We will not prove the linearization theorem in class.

**STERNBERG-GROBMAN-HARTMAN LINEARIZATION THEOREM.** If \( T(x) \) is smooth map with a hyperbolic fixed point \( x_0 \), then \( T \) is conjugated to its linearization \( DT \) near \( x_0 \).

Near the fixed point \( x_0 \), the dynamics can be computed by first going into a new coordinate system \( H^{-1}(x_0) \), applying the linear map \( A \), and undoing the change by applying \( H \).

More precisely, there exists a small disc \( D \) around \( x_0 \) and a map \( H \) in the plane such that in \( D \) the identity \( H \circ A(x) = T \circ H(x) \) holds.

**INVARINAT MANIFOLDS.** The linear equation \( x \rightarrow Ax \) has two invariant curves, the lines spanned by the eigenvectors \( v_i \) of \( A \). The conjugation defines two invariant curves \( r_i(t) = H(v_i(t)) \) through a hyperbolic fixed point. These curves are called stable and unstable manifolds of the hyperbolic fixed point. The picture shows the stable and invariant manifolds for one of the fixed points of the Henon map. The unstable manifold lies in the attractor. Note that the unstable manifold of \( T(x,y) = (1-ax^2+y, bx) \) is the stable manifold for \( T^{-1}(x,y) = (y/b, (x-1+b^2)/2b) \).

Here is the proof of the linearization theorem in its simplest case. The conjugation can actually be proven to be smooth too.

**PROOF PART I: Reduction to a global conjugation problem.**

Take first a smooth scalar function \( \phi_0(x) \), which satisfies \( \phi_0(x) = 1 \) for \( |x-x_0| > 2\varepsilon \) and \( \phi_0(x) = 0 \) for \( |x-x_0| < \varepsilon \) (see picture to the right). The map \( S = T + \phi_0(A-T) \) is equal to \( T \) for \( |x-x_0| < \varepsilon \) and equal to \( A \) for \( |x-x_0| > 2\varepsilon \). It can write \( S(x) = Ax + f(x) \), where \( f \) is a smooth map satisfying \( ||f'||_\infty < \varepsilon \) for \( \varepsilon < 0 \). Using this surgery, we can solve a global problem.

**PROOF PART II: The conjugating equation and its linearization.**

The aim is to show that \( S \) is conjugated to a map \( H(x) = x + h(x) \) to the linear map \( A \) if \( S = A + f \) if \( ||f'||_\infty \) is small enough. Remember that \( f' = DF \) is the Jacobian matrix of \( f \). The condition \( H \circ A(x) = S \circ H(x) \) can be rewritten with \( S(x) = Ax + f(x), H(x) = x + h(x) \) as

\[
(h(A(x)) - Ah(x)) = f(x) + h(x).
\]

It is an equation for the unknown map \( h \in C(X, X) \). We first consider the linearized problem

\[
(Lh)(x) := h(A(x)) - Ah(x) = f(x).
\]

**PROOF PART III: Solving the linearized problem.**

We can decompose the problem into two parts

\[
h_\pm(A(x)) - Ah_\pm(x) = f_\pm(x),
\]

where \( h = h_+ + h_- \) and \( f = f_+ + f_- \) is the decomposition satisfying \( f_- \), \( h_+ \in E^u \). The linear map on continuous functions on the plane \( U: C(X) \rightarrow C(X), A \rightarrow h(A) \) as well as its inverse \( U^{-1} \) have norm \( ||U||=||U^{-1}|| = 1 \).

We write \( A_1 = A f_+ + A f_- \). Because

\[
||U - A_1|| = ||U^{-1} \sum A_1^n U^{-n}|| \\
\leq \frac{1}{1 - \lambda}
\]

with \( \lambda = \max(||A_1||, ||A_1^-||) < 1 \), we can find \( h \) using the formula

\[
h = h_+ + h_- = (U-A_1)^{-1} f_+ + (U-A_1)^{-1} f_-. 
\]

**PROOF PART IV: Solving the nonlinear problem.**

Define \( \Phi(h)(x) = f(x+h(x)) - f(x) \). We need to solve the equation

\[
Lh = \Phi + f
\]

in for the unknown \( h \in C(X) \). The solution to this equation \( (L^{-1}\Phi - 1)h = L^{-1}f \) is

\[
h = (1 - L^{-1}f) h_0
\]

if \( 1 - L^{-1}f \) is invertible. Suitable to invertibility is that \( L^{-1}f \) is a contraction. This is indeed the case if \( \varepsilon \) is small that is if \( ||f'||_\infty \) is small:

\[
||(L^{-1}\Phi)h_0||_\infty \leq \frac{1}{1 - \lambda} \max(||h_0||_\infty, ||h_0||_\infty)
\]

**COMPUTATION OF MANIFOLDS.** The stable and unstable manifolds of a hyperbolic fixed point can be computed using power series. This calculation is due to Francesconi and Russo. To get one of the manifolds, construct a curve \( r(t) = (x(t), y(t)) \) satisfying \( r(0) = (x_0, y_0) \) and \( T(x(t), y(t)) = (1 - ax(t))^2 + y(t), bx(t)) = (x(t), y(t)) \) for all \( t \). Here \( \lambda \) is an eigenvalue of the Jacobian matrix at the fixed point. Because \( y(t) = bx(t) \), it is enough to calculate \( x(t) \). With a Taylor series \( x(t) = \sum_{n=0}^\infty a_n t^n \), the invariance condition \( 1 - ax(t)^2 + y(t) = x(t) \) or equivalently \( x(t) + ax(t)^2 - bx(\lambda^{-1}t) = 1 \) becomes

\[
\sum_{n=0}^\infty [a_n \lambda^n - ba_n \lambda^{-n} + a_n] t^n = 1.
\]

This equation allows to calculate the Taylor coefficients \( a_n \). Comparing coefficients of \( t^n \) gives \( a(a_0 a_n + a_1 a_{n-1} + ... + a_{n-1} a_1 + a_n a_0) - \lambda a_{n-1} a_n = a_n \lambda a_n \).

Once \( a_0, ..., a_{n-1} \) are given. The first coefficient \( a_0 \) is just \( x_0 \). Because \( a_1 \) satisfies \( 2a_0 a_1 - \lambda a_1 a_1 = a_1 \), it can be chosen arbitrary like \( a_1 = 1 \). For the parameters \( a = 1.4 b = 0.3 \) the unstable manifold is \( r(t) = (0.631354 + t - 0.299866 t^2 + ... , 0.189406 - 0.55946 + 0.021064 t^2 + ... ) \). The stable manifold is \( r(t) = (0.631354 + t + 0.327827 t^2 + ... , 0.189406 + 1.92374 t + 1.63796 t^2 + ...) \).

**HOMOCLINIC POINTS.** The intersection points of stable and unstable manifolds different from the fixed point itself are called homoclinic points. It has been realized already by Poincaré that the existence of homoclinic points produces a horrible mess. We will see why soon.