ABSTRACT. We look at some higher dimensional automata like the game of life or lattice gas automata. Note that 2 hours after this lecture, max time is 1111111111 Fri 18 Mar 2005 01:58:31.

HIGHER DIMENSIONAL AUTOMATA. Everything said before can be generalized to higher dimensions. Let's restrict to two dimensions. The space is \( X = \mathbb{Z}^d \). It consists of elements \( x_{n,m} \), where \( (n,m) \) are the coordinates. Define the shifts \( \sigma_1(x_{n,m}) = x_{n+1,m} \), \( \sigma_2(x_{n,m}) = x_{n,m+1} \). A continuous map on \( X \) which commutes with both \( \sigma_i \) is called a Cellular automaton. We have \( T(x_{n,m}) = \phi(x_{n,m}) \) with \( n,m \) in some finite set \( F \). The composition of two CA is a CA. A distance is defined as \( d((i,j), (i,j)) = 1/(n+1) \) if \( x_k = y_k \) for \( |k| \leq n \) and \( x_k \neq y_k \) for some \( |k| = n \), where \( ([i,j]) = [i]+[j] \).

GAME OF LIFE. One of the most famous automaton is Conway's game of life. A dead cell comes alive if and only if it has three neighbors. A live cell dies if it has less then 2 ore more than 3 neighbors.

SPECIAL SOLUTIONS. A configuration \( x \) has compact support if there are only finitely many cells which are alive. Examples of solutions with compact support are gliders, stones and blinkers.

The picture to the right shows life after a random initial condition, after having iterated for 500 iterations.

GLIDERS. Solutions which satisfy \( T^n(x) = \sigma^n(x) \) for integer \( n \) and \( v = (v_1, v_2) \) are called gliders. Gliders travel with velocity \( v/n \). If \( x \) is a glider, then \( T^n(x) \) converges to 0.

PERIODIC SOLUTIONS. If \( T^n(x) = x \), then \( x \) is called a periodic solution of \( T \). The left two configurations below show fixed points called "stones". We also see a periodic two orbits called "blinker".

THE HPP MODEL is a simple deterministic two-dimensional cellular automata designed by Hardy, Pazzis and Pomeau in 1972. Its aim is to have a simple toy model to simulate the Navier Stokes equations. The automaton has a color for each of the possible particle configurations. There can be maximally 4 particles at the same spot. One assigns a letter to each of the 16 configurations.

Particles always point away from the origin. Either there is a particle in one of the four directions, or there is not. Once can code each color with a code like \( (n, w, s, e) = (1, 1, 0, 1) \). The rules are designed such that particles move freely. For example, if if \( x_{n,m} = (0, 0, 0, 1) \) and all other nodes satisfy \( x_{i,j} = (0, 0, 0, 0) \), then \( x_{n+1,m} = (0, 0, 0, 1) \). A particle has moved from node \( (n, m) \) to node \( (n+1, m) \). If particles collide with a right angle, they will scatter as if they would pass through each other. If they hit head on, both directions change by 90 degrees.

HISTORY. Numerical treatments of ODE's and PDE's leads to CA. Examples: the heat equation \( u_t = \Delta u \) is a difference equation \( u_{i,j}^{n+1} = u_{i,j}^n + \Delta t \Delta^2 u_{i,j}^n \). If \( \Delta t \) is small, then \( u_{i,j}^{n+1} \approx u_{i,j}^n \) for \( \Delta t \) small enough. The discretisation is a CA with an alphabet of 1 letter. The computing accuracy is \( \Delta t \). Different methods for PDEs were used since a long time, at least since 1940 (K.P. Richtmayer), and research on it exploited during WW2 and when the first computers appeared (i.e., the first electronic computer ENIAC in 1944). John von Neumann seemed have introduced CA in these years. Ulam claims to have found CAs first in "Adventures of a Mathematician" p.285: "my own simple minded model". 1960 Turing machines are shown to be able to do all computations. A Turing machine with a states and a tape alphabet of \( k \) symbols is a special cellular automata with an alphabet of \( k \times 2 \) letters.

1982 Idealized models of biological systems were studied using CA. Ulam and Von Neumann called this "nearest neighbor connected cellular spaces". Source: From Cardinals to Chaos, Ed: Steve Christenson, Cambridge University Press.

