THE SBKP MAP. For $|k| < 1$, let us call the map

\[ T(x,y) = (2x + 4 \cdot \arg(1 + k \cdot e^{-xy}) - y, x) \]

on the torus the **Suri-Bobenko-Kutz Pinkall map**. It had been found by Bobenko-Kutz. Pinkall and independently by Suris. Even so the map uses complex numbers for its definition, it is real. The argument \( \arg(z) \) of a complex number \( z = x + iy = r \cos(\alpha) + ir \sin(\alpha) = re^{i\alpha} \) is defined as the angle \( \alpha \).

**THEOREM.** The SBKP map is integrable.

**PROOF.** The function

\[ F(x,y) = 2(\cos(x) + \cos(y)) + k \cdot \cos(x + y) + k^{-1} \cdot \cos(x - y) \]

is an integral. It is not easy to verify that. Don’t ask how \( F \) was found!

**THE COHEN-COLLINE-DE VERDIERE MAP.** The map

\[ T(x,y) = (\sqrt{x^2 + \epsilon^2} - y, x) \]

in the plane is called the **Cohen-Colline-de Verdiere map**. By rescaling coordinates in \( \mathbb{R}^2 \), we can assume \( \epsilon = 0 \) or \( \epsilon = 1 \). For \( \epsilon = 0 \), the map has the form

\[ T(x,y) = (|x| - y, x) \, . \]

We call it the **Knuth** map.

**THEOREM (KNUTH)** The Knuth map is integrable.

**PROOF.** We check that \( T^{14} = Id \). Note that the map is piecewise linear, we only have to look at the orbits of the \( x \) axes to understand the entire picture. Actually, every orbit is periodic with period 1, 3 or 9.

**LEMMA.** A map in the plane for which there exists \( n \) such that \( T^n(x,y) = (x,y) \), must be integrable.

**PROOF.** Take \( f(x,y) = y \) for example. Then \( F(x,y) = \sum_{i=0}^{n-1} f(T^i(x,y)) \) is an integral.

If we apply this lemma to the Knuth map, we get an explicit integral

\[ F(x,y) = y + |y - x| + |x - |y - x|| + |y - |x - y|| + |x - |y - |x|| + |y - |x - |y|| \, . \]

The level curves of this function are shown in the graphics above. For every value \( c > 0 \) the level set \( F(x,y) = c \) is a closed gingerman shaped curve on \( T \) conjugated to a rotation by an angle \( 1/9 \).


**INTEGRABLE OR NOT?** Let’s look at the case \( \epsilon = 1 \), where

\[ T(x,y) = \left( \sqrt{x^2 + 1} - y, x \right) \]

All orbits seem all to lie on invariant curves. The map looks integrable. It had been communicated to me by M. Rychlik in 1998, that numerical experiments by John Hubbard revealed a hyperbolic periodic orbit of period 14: \( (x,y) = (u, u) \) with \( u = 1.54871181145059 \). The largest eigenvalue of \( DT^{14}(x,y) \) is \( \lambda = 1.012275907 \). The existence of a hyperbolic point of such a period makes integrability unlikely since homoclinic points might exist, but it is not impossible. It is difficult to find other hyperbolic periodic points. An other indication for non-integrability is a result of Rychlik and Torgenson who have shown that this map has no integral given by algebraic functions.
HOW TO FIND AN INTEGRAL?

If we know a map is integrable, we could recover the invariant function \( F \) by taking \( f(x, y) = y \) and defining \( F(x, y) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x, y)) \).

This invariant function is called the time average along the orbit. In the case of nonintegrability, this function is constant on complicated sets or even be infinite on some part of the plane. If the map is integrable with a nice analytic function, one could expect the integral to found using time averages.

THE McMILLAN MAP \( T \) is an other example of an integrable map, where \( k \) is a parameter. It is called the McMillan map and has the integral

\[
|F(x, y)| = x^2 + y^2 + x^2y^2 - 2kxy.
\]

It is especially interesting to study because \( T \) is a rational function, a fraction of two polynomials. I don’t think, one has a complete list of all integrable rational maps in the plane.

WHAT HAPPENS CLOSE TO THE INTEGRABLE CASE? In general, integrability gets lost when making small changes to an integrable map. For example, the Standard map \( T(x, y) = (2x - y + \sin(x), x) \) can for small \( \epsilon \) be considered as a perturbation of the integrable map \( T(x, y) = (2x - y, x) \) which has the integral \( F(x, y) = x - y \). A study of the stable and unstable manifolds of the hyperbolic fixed point \((0, 0)\) shows that they intersect transversely for small \( \epsilon \). One usually studies the map in another form. Because \( H(x, y) = (-x, y - x) = H^{-1}(x, y) \) satisfies \( H(S(H(x, y))) \), where \( T(x, y) = (2x - y + \sin(x), x) \) and \( S(x, y) = (x + y + \sin(x), y + \sin(x)) \), we can look also at the map \( S \) instead. This map has the integral \( F(x, y) = y \) for \( \epsilon = 0 \) and the invariant curves are horizontal.

KAM. Near integrable maps, remnants of integrability still exist. These traces of integrability persist in the form of smooth invariant curves which are now called KAM curves. The acronym KAM stands for Kolmogorov-Arnold-Moser. The proof that invariant curves persist after the perturbation is not easy. To find an invariant curve on which the map is conjugated to an irrational rotation with angle \( \alpha \), we need to find a periodic function \( q(x) \) such that \( q_n = q(n\alpha) \) satisfies the nonlinear recursion \( q_{n+1} - 2q_n + q_{n-1} = \epsilon \sin(q_n) \). This means

\[
q(x + \alpha) - 2q(x) + q(x) = \epsilon \sin(q).
\]

Naively, one could try to find \( q \) using the implicit function theorem: if one could invert the linear map \( L(q) = q(x + \alpha) - 2q(x) + q(x) \).

SMALL DIVISORS. Let’s look at this inversion problem If \( q(x) = \sum a_n e^{inx} \) is the Fourier series of \( q \), then \( Lq(x) = \sum a_n e^{inx} - 2e^{inx}e^{inx} \). If \( L(q) = p = \sum d_n e^{inx} \), then

\[
q = L^{-1}p = \sum a_n e^{inx} - \frac{1}{2} + e^{-inx}e^{inx} = \sum b_n \frac{2}{\cos(n\alpha) - 1} e^{inx}.
\]

You see the appearance of small divisors \( \frac{2}{\cos(n\alpha) - 1} \). In order that the Fourier series of the inverse converges, one needs \( \alpha \) to be far away from rational numbers. Such numbers are called Diophantine numbers. Even so, one is able to invert \( L \) in certain cases, the map \( L \) is not invertible as required for the implicit function theorem. One needs a so called hard implicit function theorem.