11. homework set

11.1 Describe the geodesic flow on the one-sheeted hyperboloid. How does a typical geodesic look like. Find one periodic geodesic.

11.2 When you empty a paper towel roll you obtain a cylinder. On this cylinder you find a spiral curve. Prove first that this curve is a geodesic. Assume the cylinder is \( x^2 + y^2 = 1, 0 \leq z \leq 1 \). We play surface billiard: the geodesic curve is reflected at the boundaries \( z = 0, z = 1 \). Find the return map \( T(\theta, \phi) = (\theta_1, \phi_1) \), where \((x, y, z) = (\cos(\theta), \sin(\theta), 0)\) and \( \phi \) is the impact angle.

Optional: From what you know about billiards, can you cut away part of the cylinder to get a chaotic surface billiard?

11.3 For a surface of revolution which is symmetric to the \( xy \) plane and for which \( r(z) \to 0 \) for \( |z| \to \infty \) we define a Poincare map for the geodesic flow. Start with a point \( x \) on the surface in the \( xy \) plane and a unit vector \( v \), follow the geodesic flow along the surface until it comes back to the surface. This defines return map \( T(\theta, \phi) \) on the annulus \((\theta, \phi) \in T \times [0, \pi]\). Describe this map in the case of the sphere.

11.4 We play surface billiard in the triangular surface obtained by intersecting the unit sphere with the first octant in space. Prove that this billiard is integrable.

Hint. Remember how you analyzed billiards in the rectangle? A similar idea applies here.

11.5 We play surface billiard in the half cone \( x^2 + y^2 = 1 - z^2 \). Prove that there are orbits which never close.

Hint. Similar than for the flat torus or the cylinder one can find geodesics by cutting up the surface and flattening it.

11.6 (optional) Can you prove the statement made in class that on a flat torus, the wave front \( K_t(x) \) of a point becomes dense on the torus in the sense, given \( \epsilon > 0 \), there is a \( s \) such that \( K_t \) intersects every disc of radius \( \epsilon \) for \( t > s \).

Remark: We do not know whether this statement stays true, if you allow the torus to be bumpy. Nor do we know whether the caustic \( C_t(x) \) becomes dense in general. \( (C_t(x) \) is the empty set for the flat torus).