THE CANTOR SET. The Cantor set is constructed recursively by dividing the interval $[0,1]$ into 3 equal intervals and cutting away the middle one repeating this procedure with each of the remaining intervals etc. At the $k$th step, we need $2^k$ intervals of length $1/3^k$ to cover the set. The s-volume $h_{s=1}(X)$ of accuracy $1/3^k$ is $8^k(1/3)^{sk}$ which goes to 0 for $k$ approaching infinity if $s$ is smaller than $d = \log(8)/\log(3)$ and diverges for $s$ bigger than $d$. The dimension of the carpet is $d = \log(8)/\log(3) = 1.893$ a number between 1 and 2. It is a fractal.

SHIRPINSKI CARPET. The Shirpinski carpet is constructed recursively by dividing a square in the plane into 9 equal squares and cutting away the middle one, repeating this procedure with each of the remaining squares etc. At the $k$th step, we need $8^k$ squares of length $1/3^k$ to cover the carpet. The s-volume $h_{s=1}(X)$ of accuracy $1/3^k$ is $8^k(1/3)^{sk}$ which goes to 0 for $k$ approaching infinity if $s$ is smaller than $d = \log(8)/\log(3)$ and diverges for $s$ bigger than $d$. The dimension of the carpet is $d = \log(8)/\log(3) = 1.893$ a number between 1 and 2.

MENGER SPONGE. The three-dimensional analogue of the Cantor set in one dimensions and the Shirpinski carpet. One starts with a cube, divides it into 27 pieces, then cuts away the middle third along each axes. It is your task to compute the dimension. Note that the faces of the Menger sponge are decorated by Shirpinski Carpets.

THE PROBLEMS OF THE DEFINITION. If one takes the above definition, then the dimension of the set of rational numbers in the interval $[0,1]$ is equal to 1. A better definition, the Hausdorff dimension is needed. We include that definition below but it is a bit more complicated. The problem with the box counting dimension is that the size of the cubes should be allowed to vary. This refinement is similar to the change from the Riemann integral to the Lebesgue integral.

HAUSDORFF MEASURE. Let $(X,d)$ be a metric space. Denote by $|A| = \sup_{U,r\in\mathcal{U}} d(x,y)$ the diameter of a subset $A$. Define for $\epsilon > 0, s > 0$ $h_s(\epsilon) = \inf_{U_{\epsilon}\in\mathcal{U}_{\epsilon}} \sum_{U_i} |U_i|^s$, where $U_{\epsilon}$ runs over all countable open covers of $A$ with diameter $< \epsilon$. Such covers are also called $\epsilon$-covers. The limit $h_s(\epsilon) = \lim_{\epsilon\to0} h_s(\epsilon)$ is called the s-dimensional Hausdorff measure of the set $A$. Note that this limit exists in $[0,\infty]$ (it can be $\infty$), because $\epsilon \mapsto h_s(\epsilon)$ is increasing for $\epsilon \to 0$.

LEMMA: If $h_s(\epsilon) < \infty$, then $h_s(\epsilon) = 0$ for all $t > s$. Take $\epsilon > 0$ and assume $(U_j)_{j\in\mathbb{N}}$ is an open $\epsilon$-cover of $A$. Then $h_s(\epsilon) \leq \sum_j |U_j|^s \leq \epsilon^{-s} \sum_j |U_j|^s$. Taking the infimum over all coverings gives $h_s(\epsilon) \leq \epsilon^{-s} \cdot h_s(\epsilon)$. In the limit $\epsilon \to 0$, we obtain from $h_s(\epsilon) < \infty$ that $h_s(\epsilon) = 0$.
HAUSDORFF DIMENSION.
Either there exists a number \( \dim_H(A) \geq 0 \) such that
\[
\begin{align*}
  s < \dim_H(A) & \Rightarrow h^s(A) = \infty, \\
  s > \dim_H(A) & \Rightarrow h^s(A) = 0
\end{align*}
\]
or for all \( s \geq 0 \), \( h^s(A) = 0 \). In the latter case, one defines \( \dim_H(A) = \infty \).
The number \( \dim_H(A) \in [0, \infty] \) is called the Hausdorff dimension of \( A \).

HISTORY.
The Cantor set is named after George Cantor (1845-1918), who was putting the foundations of set theory. Ian Stewart writes in “Does God Play Dice”, 1989 p. 121: “The appropriate object is known as the Cantor set, because it was discovered by Henry Smith in 1875. The founder of set theory, George Cantor, used Smith’s invention in 1883. Let’s fact it, ‘Smith set’ isn’t very impressive, is it?”

The Hausdorff dimension has been introduced in 1919 by Felix Hausdorff (1868-1942).

The Sierpinski carpet was studied by Waclaw Sierpinski in 1916. He proved that it is universal for all one dimensional compact objects in the plane. This means that if you draw a curve in the plane which is contained in some finite box, however complicated it might be and with how many self-intersections you want, there is always a part of the Sierpinski carpet which is topologically equivalent to this curve.

This might not look so surprising but this result is not true for the Sierpinski gasket. The Menger Sponge was studied by Klaus Menger in 1926. He showed that it is universal for all one dimensional objects in space. This means whatever complicated curve you draw in space, you find a part of the Menger sponge, which is topologically equivalent to it.

SELF-SIMILARITY. The computation of the dimension in the example objects was easy because they are self-similar. A part of the object is when suitably scaled equivalent to the object. We will see more about this when we look at iterated function systems. To measure or estimate the dimension of an arbitrary object, one has to count squares. As an illustration of fractals in nature, one often takes coast lines. A rough estimate of the coast of Massachusetts leads to a dimension 1.3.

HAUSDORFF DIMENSION.
Either there exists a number \( \dim_H(A) \geq 0 \) such that
\[
\begin{align*}
  s < \dim_H(A) & \Rightarrow h^s(A) = \infty, \\
  s > \dim_H(A) & \Rightarrow h^s(A) = 0
\end{align*}
\]
or for all \( s \geq 0 \), \( h^s(A) = 0 \). In the latter case, one defines \( \dim_H(A) = \infty \).
The number \( \dim_H(A) \in [0, \infty] \) is called the Hausdorff dimension of \( A \).

FRACTAL. A fractal is a subset of a metric space which has finite non-integer Hausdorff dimension.

The Hausdorff dimension is in general difficult to calculate numerically. The central difficulty is to determine the infimum over \( \sum |U_i|^s \), where \( U = \{U_i\} \) is an \( \epsilon \)-cover of \( A \). The box-counting dimension simplifies this problem by replacing arbitrary covers by sphere covers and so to replace the terms \( |U_i|^s \) by \( \epsilon^s \). The prize one has to pay is that one can no more measure all bounded sets like this. In general, the upper and lower limits differ.

UPPER AND LOWER CAPACITY. Given a compact set \( A \subset X \). Define for \( \epsilon > 0 \), \( N_\epsilon(A) \) as the smallest number of sets of diameter \( \epsilon \) which cover \( A \). By compactness, this is finite. Define the upper capacity
\[
\dim_{\text{up}}(A) = \limsup_{\epsilon \to 0} \frac{\log(N_\epsilon(A))}{\log(\epsilon)}
\]
and analogous the lower capacity \( \dim_{\text{low}}(A) \), where \( \limsup \) is replaced with \( \liminf \). If the lower and upper capacities coincide, the value \( \dim_{\text{up}}(A) \) is called box counting dimension of \( A \).

CAPACITY DIMENSION. If the lower and upper capacity are the same, one calls it the capacity dimension.

BOX COUNTING DIMENSION. Cover \( \mathbb{R}^n \) by closed square boxes of side length \( 2^{-k} \). and let \( M_k(A) \) be the number of such boxes which intersect \( A \). Define the box counting dimension
\[
\dim_{\text{box}}(A) = \lim_{k \to \infty} \frac{\log(M_k(A))}{\log(2^k)}.
\]
If the capacity dimension exists, then it is equal to the box counting dimension.

PROOF: Any set of diameter \( 2^{-k} \) can intersect at most \( 2^k \) grid boxes. On the other hand, any box of side \( 2^{-k} \) has diameter smaller than \( 2^{-k+1} \). There exists therefore a constant \( C \) such that
\[
C^{-1} \cdot M_k(A) \leq N_{2^{-k+1}}(A) \leq C \cdot M_k(A).
\]
Therefore
\[
\lim_{k \to \infty} \frac{\log(M_k(A))}{\log(2^k)} = \lim_{k \to \infty} \frac{\log(N_{2^{-k+1}}(A))}{\log(2^k)}.
\]

The name “fractal” had been introduced only much later by Benoit Mandelbrot (1924-) in 1975.

Abram Besicovitch, around 1930, worked out an extensive theory for sets with finite Hausdorff measure.

The Sierpinski carpet was studied by Waclaw Sierpinski in 1916. He proved that it is universal for all one dimensional compact objects in the plane. This means that if you draw a curve in the plane which is contained in some finite box, however complicated it might be and with how many self-intersections you want, there is always a part of the Sierpinski carpet which is topologically equivalent to this curve.

The Hausdorff dimension has been introduced in 1910 by Felix Hausdorff (1868-1942).

Ordinary measure. The measure of a set \( A \subset \mathbb{R}^n \) is given by
\[
\mu(A) = \int_A \, d\nu,
\]
where \( \nu \) is the standard n-dimensional Lebesgue measure.