2/2/04 WHAT ARE DYNAMICAL SYSTEMS? Math118, O.Knill

ABSTRACT. We discuss in this lecture, what dynamical systems are and where the subject is located within mathematics.

A FIRST DEFINITION.
The theory of dynamical systems deals with the evolution of systems. It describes processes in motion, tries to predict the future of these systems or processes and understand the limitations of these predictions.

RELEVANCE OF DYNAMICAL SYSTEMS.
To see that dynamical systems are relevant, one has just to look at a few news stories which broke during the last few weeks:
- Tsunami damage prediction
- Meteor path computation
- Currents in the sea
- Landing of the Cassini probe on Titan

A FANCY DEFINITION.
Mathematically, any semigroup \( G \) acting on a set is a dynamical system. A semigroup \( (G, \ast) \) is a set \( G \) on which we can add two elements together and where the associativity law \((x \ast y) \ast z = x \ast (y \ast z)\) holds. The action is defined by a collection of maps \(T_t\) on \(G\). It is assumed that \(T_{s+t} = T_s \circ T_t\), where \(\ast\) is the operation on \(G\) (usually addition) and \(\circ\) is the composition of maps.

CLASSES OF DYNAMICAL SYSTEMS:

<table>
<thead>
<tr>
<th>Time ( G ) (semigroup)</th>
<th>Action</th>
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</thead>
<tbody>
<tr>
<td>Natural numbers ((\mathbb{N}, +))</td>
<td>Maps</td>
</tr>
<tr>
<td>Integers ((\mathbb{Z}, +))</td>
<td>Invertible maps</td>
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<tr>
<td>Positive real numbers ((\mathbb{R}^+, +))</td>
<td>Semiflows (some PDE’s)</td>
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<td>Any group ((G, \ast))</td>
<td>Flows (Differential equations)</td>
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<tr>
<td>Lattice ((\mathbb{Z}^n, +))</td>
<td>Lattice gases, Spin systems</td>
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<tr>
<td>Euclidean space ((\mathbb{R}^n, +))</td>
<td>Iterated function systems</td>
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TWO IMPORTANT CASES OF ONE DIMENSIONAL TIME. We mention the general definition to stress that the ideas developed for one dimensional time generalize to other situations. Because physical time is one dimensional, the important cases for us are definitely discrete and continuous dynamical systems:

- dynamics of maps defined by transformations
- dynamics of flows defined by differential equations

DYNAMICAL SYSTEMS AND THE REST OF MATH. All areas of mathematics are linked together in some way or another. Intersections of fields like algebraic topology, geometric measure theory, geometry of numbers or algebraic number theory can be considered full blown independent subjects. The theory of dynamical systems has relations with all other main fields and intersections typically form subfields of both.

EXAMPLES OF INTERSECTIONS OF DYNAMICS WITH OTHER FIELDS:
- Link with **algebra**: group theorists often look at the action of the group on itself. The action of the group on vector spaces defines a field called representation theory.
- Link with **measure theory**: in ergodic theory one studies a map \(T\) on a measure space \((X, \mu)\). Measure theory is one foundation of ergodic theory.
- Link with **analysis**: the study of partial differential equations or functional analysis as well as complex analysis or potential theory.
- Link with **topology**: the Poincare conjecture states that every compact three dimensional simply connected manifold is a sphere. The problem is currently attacked using a dynamical system on the space of all surfaces which is called the Ricci flow.

- Link with **geometry**: Kleins Erlanger program attempted to classify geometries by its symmetry groups. For example, the group of projective transformations on a projective space. A concrete dynamical system in geometry is the geodesic flow. An other connection is the relations of partial differential equations with intrinsic geometric properties of the space.

- Link with **probability theory**: sequences of independent random variables can be obtained using dynamical systems. For example, with \(T(x) = 2x \mod 1\) and with the function \(f\) which is equal to 1 on \([0, 1/2]\) and equal to 0 on \([1/2, 1]\), \(f(T^n(x))\) are independent random variables for most \(x\).

- Link with **logic**: logical deductions in a proof or doing computations can be modeled as dynamical systems. Because every computation by a Turing machine can be realized as a dynamical system, there are fundamental limitations, what a dynamical system can compute and what not.

- Link with **number theory**: some problems in the theory of Diophantine approximations can be seen as problems in dynamics. For example, if you take a curve in the plane and look at the sequence of distances to nearest lattice points, this defines a dynamical system.

- A final link: a **category** \(X\) of mathematical objects has a semigroup \(G\) of homomorphisms acting on it (topological spaces have continuous maps, sets have arbitrary maps, groups, rings fields or algebras have homomorphisms, measure spaces have measurable maps). We can view each of these categories as a dynamical system. One can even include the category of dynamical systems with suitable homomorphisms. But this viewpoint is not a very useful in itself.