POINCARÉS RECURRENCE THEOREM. Area preservation allows to make a statement about recurrence of area-preserving map defined on a T invariant subset in the plane. For example, X could be the annulus \( R/Z \times [-1, 1] \) and T could be a billiard map. For every set Y of positive area \( [Y] \), there exists n such that \( T^n(Y) \cap Y \) has positive area.

\[ \text{THEOREM. (Birkhoff)} \text{ There are no invariant curves of } T, \text{ if and only if there exists a glancing orbit.} \]

**PROOF.** If there is an invariant curve, there is trivially no glancing orbits because the regions on both sides of the curve are left invariant. Assume now there is no glancing orbit. This means there is an \( \epsilon > 0 \) such that for all \( y_j > 1 + \epsilon \). Consider the region \( Y = \{ y < 1 - \epsilon \} \). The set \( \bigcup Y \) is a T-invariant set which does not intersect \( \{ y > 1 - \epsilon \} \). The boundary of this curve is an invariant curve. (One actually knows that such a curve must be the graph of a Lipschitz continuous function.)

\[ \text{THE JACOBEAN. Let } \kappa, \text{ denote the curvature at the impact point and angle } \theta, \text{ the impact angle and let } l, \text{ the length of the path from the impact point } x_{i-1} \text{ to the impact point } x_i. \text{ The following formula is well known in geometrical optics and used elsewhere in the billiard literature like in the book of Kozlov-Treschev.} \]

**LEMMA.** There are coordinates for which the Jacobean \( DT(x, y) \) of the billiard map has the form

\[
B_i = \begin{bmatrix}
\frac{1}{2\kappa_i} & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & l_i \\
0 & 1
\end{bmatrix}
\]

Remark: This is the composition of the Jacobean belonging to the translation and the Jacobean belonging to the reflection at the wall. The value \( \kappa_i = \frac{\text{sin}(\theta_i)}{l_i} \) is the length of the billiard ball in the circle on the normal to the reflection point which is tangent to the table and has radius \( 1/(2\kappa_i) \).

**PROOF OF THE JACOBEAN FORMULA.** The formula can be derived geometrically. Instead, we find an algebraic derivation from the Euler equations. It is still a bit messy.

We use the notation \( h_1, h_2 \) for the first and second partial derivative with respect to the first variable and similar \( h_2 \) for the mixed partial derivative. The billiard map \( S : \begin{bmatrix} x_i \\ x_{i+1} \end{bmatrix} \rightarrow \begin{bmatrix} x_{i+1} \\ x_i \end{bmatrix} \) is equivalent to the second order recursion

\[
h_1(x_i, x_{i+1}) + h_2(x_{i-1}, x_i) = 0.
\]

Differentiation of these Euler equation with respect to \( x_i, x_{i-1} \) gives \( \partial_x A_i = -h_2(x_i, x_{i+1}) \) and \( b_i = h_2(x_i, x_{i+1}) + h_2(x_{i-1}, x_i) \).

\[
\text{The Jacobian of } S \text{ is } dS = \begin{bmatrix}
-h_2/a_i & -a_{i-1}/a_i \\
0 & 1
\end{bmatrix}.
\]

With a first coordinate transformation \( F_i = \begin{bmatrix} a_{i-1}^{-1} & 0 \\
0 & 1 \end{bmatrix} \) we can achieve that the determinant is 1:

\[
E_i^{-1}dSF_{i-1} = A_i = \begin{bmatrix} 1/h_i & -a_i^{-1} \\
0 & 1 
\end{bmatrix}.
\]

Geometrically, we have

\[
a_i = \frac{\text{sin}(\theta_i)}{l_i}, \quad b_i = \frac{\text{sin}^2(\theta_i)}{l_i} \left( 1 + \frac{1}{l_i} - 2\text{sin}(\theta_i)/l_i \right),
\]

where \( l_i = h(x_i, x_{i+1}) \) are the lengths of the secants, \( \theta_i = \theta(x_i, x_{i+1}) \) and \( a_i = a(x_i) \) are the curvatures at the reflection points. Plugging this in the Jacobean gives with \( G_i = \begin{bmatrix} 0 & -\text{sin}(\theta_i) \\
1/\text{sin}(\theta_i) & \text{sin}(\theta_i)/l_i \end{bmatrix} \) the new Jacobean

\[
G_i^{-1}A_iG_{i-1} = \begin{bmatrix}
\frac{1}{2\kappa_i} & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & l_i \\
0 & 1
\end{bmatrix}.
\]
STABILITY OF PERIOD 2 ORBITS. Having the Jacobean given in geometric terms allows to see, whether periodic orbits are stable or not. Inspection of the trace of $B_i B_j$ (a matrix which is similar to the Jacobean of $T^2$) and so has the same trace) shows:

**LEMMA** Assume $r_i$ are the radii of curvature at the impact points. Assume $r_1 < r_2$. If $l > r_1 + r_2$ or $r_1 < l < r_2$, then the periodic orbit of period 2 is hyperbolic. If $l > r_2$ or $l < r_1 + r_2$, it is elliptic.

The fastest verification of the lemma is to run a line of Mathematica which gives the trace of the product of the four matrices. For example, the long axis of a non-circular ellipsoid is a hyperbolic periodic point. The short axis is an elliptic periodic point.

CURVATURE. If $r(s)$ is a curve in the plane parametrized by arc-length, then the curvature $\kappa(t) = |r'(s)|$. If $r(t)$ is the curve given by an arbitrary parameterization, define the unit tangent vector $\hat{T}(t) = r'(t)/|r'(t)|$. We get the curvature $\kappa(t) = \hat{T}(t)||r''(t)||$. The function $\rho(t) = 1/\kappa(t)$ is called the radius of curvature. With the crossed product $(a, b) \times (c, d) = ad - bc$ in two dimensions, we have a more convenient formula $\rho(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^2}$.

ROLE OF CURVATURE. The curvature of the table plays an important role for the billiard dynamics. Here are some known results:

- Mather has shown that if the table has a flat point, this is a point at which the curvature vanishes like at 4 points of $x^2 + y^2 = 1$, then the billiard map $T$ has no invariant curve at all.
- Lazutkin and Donadu have proven using KAM theory that for a smooth billiard table with positive curvature everywhere, there always are "whisper galleries" near the table boundary.
- From Andrea Hubacher (who had obtained this result as an undergraduate student at ETH) is the result that a discontinuity in the curvature of the table does not allow caustics near the boundary. For example, tables obtained by the string construction at a triangle (see homework) do not allow invariant curves near the boundary.
- It is easy to see that billiards for which the table has negative curvature everywhere, the Lyapunov exponent is positive. The Matrices $B_i$ have then positive entries as we will just see.

POSITIVE MATRICES. If we multiply positive matrices with each other, the norm of the product grows exponentially.

**LEMMA** If $\det(A(x)) = 1$ for all $x$ and $\det(A_{ij}(x)) \geq \epsilon > 0$, then the Lyapunov exponent $\lambda(x) = \lim_{n \to \infty} \log||A(T^{n-1}x) A(T^{n-2}x) \ldots A(x)||$ satisfies $\lambda(A) \geq \frac{1}{2} \log(1 + 2 \epsilon^2)$.

**PROOF** (Wojtkowski). Define the function $F$ on pairs of vectors by $v = (v_1, v_2) \mapsto F(v) = (v_1, -v_2)^{1/2}$. For a matrix $B$ with determinant 1 satisfying $|B_{ij}| = \epsilon$, define $\rho(B) = \inf_{F(v) = 1} F(Bv)$. 

1. Given a $2 \times 2$-matrix $A$ satisfying $|A_{ij}| \geq \epsilon$. Then $\rho(A) \geq (1 + 2 \epsilon^2)^{1/2}$. Proof: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $w = (w_1, w_2)$ and $F(w) = (w_1 w_2)^{1/2}$, then $F(Aw) = (caw + bdu)^{1/2}(cau + dww)^{1/2} \geq (ad - bc + 2\epsilon^2)^{1/2} \geq (1 + 2\epsilon^2)^{1/2}$.
2. $|F(B)| \geq \rho(B)$. Proof: Take $v = (1, 1)$. Then $|F(B)| \geq \frac{4w_1}{\epsilon} \geq \frac{2\epsilon w}{\epsilon} \geq \rho(A)$.
3. $\rho(A) = \inf_{F(v) = 1} F(Bv) \geq \inf_{F(v) = 1} F(Bv) \geq 2 \epsilon w / \epsilon$.
4. We get from (ii), (iii), (iv) that $\frac{1}{2} \log||A^n(x)|| \geq \frac{1}{2} \log(\rho(A(T^{n-1}x) \ldots A(x)) \geq \frac{1}{2} \log((1 + 2 \epsilon^2)^{1/2})$.

CLASSES OF CHAOTIC BILLIARDS. Remember that $g = \frac{\sin \theta}{\cos \theta}$, and $\ell$ is the length of the trajectory.

**THEOREM** (Wojtkowski) Assume, a piecewise smooth convex table has the property that for any pair of points $x, x'$ on the non-flat parts of the curve $2y + 2y' \leq l(x, x')$, with strict inequality on a set of positive measure, then the billiard map $T$ has positive Lyapunov exponents on a set of positive measure.

**PROOF** The Jacobian matrix is conjugated to $B_i(x) B_j(x)$. A vector $v = (1, f)$ is mapped by the matrix $B_i(x)$ to the vector $(1, f + l(x))$. This vector is then mapped by $B_i(x)$ to the vector 

$$(1 - (f + l(x))/2g(Tx), f + l(x))$$

which is after a rescaling of length equal to the vector 

$$\frac{(1 - (f + l(x)))(g(Tx))}{2g(Tx) - f - l(x)}.$$

If we don’t care about the length of the vector, the map $v = B(x)v$ is determined by the map 

$$K : f \mapsto f + l = \frac{1}{1/f + f - l(g(Tx))} - \frac{f + l(2g(Tx))}{2g(Tx)}.$$

At each point $x \in X$, we define a basis given by $e_2(x) = (1, 0)$ and $e_1(x) = (1, -g(x))$.

Claim: Assume $2g(x) + 2g(Tx) \leq l(x)$ with inequality on a set of positive measure. In this basis, the matrix $B(x)$ is positive and there exists a set of the measure, where $B(x)e_i \geq \epsilon$ for some $\epsilon > 0$ so that we can apply the previous lemma on positive matrices.

Proof. We have to show that the map $K$ maps the interval $[0, -2g(x)]$ into the interval $[0, -2g(Tx)]$ and into its interior for a set of positive measure because:

$$K(0) = \frac{l(2g(Tx))}{2g(Tx)} - l([0, -2g(Tx)]) \geq -2g(Tx),$$

$$K(-2g(x)) = \frac{-2g(x)}{2g(Tx) + 2g(x)} - l([-2g(x)]) \leq 0.$$

BUNIMOVICH STADIUM. A famous example is the stadium, where two half circles are joined by straight lines. An other example is the rounded square.

For these billiards, one knows actually much more. They are ergodic and chaotic in the sense of Devaney, a notion we have met earlier in this course. The prove of ergodicity is not so easy. One has to analyze some stable and unstable manifolds and verify that they are dense.

OPEN PROBLEMS. The following problems are open mathematical problems. The first two problems probably go back to Poincaré. The third problem is an old problem in smooth ergodic theory. The difficulty of that problem is that for a smooth convex billiards, there are lots of invariant curves and also lots of elliptic periodic orbits consequently, the chaotic regions are mingled well with the stable regions and the techniques described in this handout do not work.

1) Are periodic orbits dense in the annulus for a general smooth Birkhoff billiard?
2) Is the total measure ("area") of the periodic orbits always zero in the annulus? One knows it for period 3 (Rychlik).
3) Does there exist a smooth convex billiards with positive Lyapunov exponents on a set of positive measure (="area"="probability")?