WAVE FRONTS AND CAUSTICS

THE MIRROR EQUATION. If \( P \) and \( Q \) are successive points on a caustic for a geodesic ray which is reflected at the boundary point \( M \) with curvature \( \kappa \) and impact angle \( \theta \), then \( f = |P - M| \) and \( e = |Q - M| \) satisfy

\[
\frac{1}{f} + \frac{1}{e} = \frac{1}{\kappa} \tan \theta.
\]

PROOF. The change of the incoming angle \( d\theta_{in} \) and the outgoing ray \( d\theta_{out} \) is related by \( d\theta_{out} = 2d\theta_{in} + d\theta_{1} \). The claim follows from \( d\theta_{1} = \frac{1}{\kappa} \tan \theta / f, d\theta_{2} = \sin \theta / e \).

Interpretation: If \( P = x \) is a point, then \( Q \) is a point of the differential geometrical caustic \( C_{x} \) of the point \( x \).

EXAMPLES. If you light a flashlight at a given point \( x \), the wave front \( K_{s}(t) \) becomes dense on the surface for every point \( x \). The caustic is empty. The picture to the right shows the wave front on the flat torus at time 3.

ROUND SPHERE. The wave front \( K_{s}(t) \) is a circle or a point at all times. In the case of the flat torus, the caustic is empty, in the case of the sphere, the caustic \( C_{x} \) consists of two points, \( x \) and the antipole \( S(x) \) of \( x \).

CAUSTIC FLAT CASE. Let \( \gamma : r(\alpha) = (x(\alpha), y(\alpha)) \) be a curve in the flat plane and let \( n(\alpha) = (-y(\alpha), x(\alpha)) \) be the normal vector to the curve and \( p(\alpha) = 1/\|n(\alpha)\| = 1/\|r(\alpha)\| \). Then \( K_{s}(t, \alpha) = r(\alpha) + tn(\alpha)p(\alpha) = (x(\alpha) - tp(\alpha), y(\alpha)) + te(\alpha)p(\alpha) \) so that \( DK_{s}(t, \alpha) = \left[ n(\alpha)p(\alpha) \right] r'(\alpha) + tn'(\alpha)p(\alpha) + tn(\alpha)p(\alpha) \) = \( \left[ n(\alpha)p(\alpha) \right] r'(\alpha) + tn'(\alpha)p(\alpha) \) using \( \det(\alpha, \beta, \gamma) = \det(\alpha, \beta) \). The caustic of the curve \( \gamma \) is called the evolute of the curve.

EXAMPLE. Locally, we can represent a plane curve as a graph \((x, f(x))\). The wave front \( W(t, x) = (x, f(x)) + t((-f'(x), 1)/\sqrt{1 + f'(x)^2}) \) has the caustic

\[
\{(t, x) : f'(x) = \sqrt{1 + f'(x)^2}\}.
\]

For example, for \( f(x) = x^2 \), we have \( \{W(1 + 4x^2)^{1/2}/2, x) = \{(x, 1/2 + 3x^2) \) which is essentially the graph of \( y = x^{2/3} \). For \( f(x) = x^3 \), we have \( \{W(1 + 16x^2)^{1/2}/(12x^2), x) = \{(2x/3 - 16x^2/3, 7x^4/3 + x^{-2}/12) \} \).

CAUSTICS OF BILLIARDS. The word “caustic” has different meaning in billiards and in differential geometry. Caustics can be defined for any family of light rays. In differential geometry, one looks at all the light rays which are emitted at one point or all light rays emitted orthogonally to a given curve. If we look at all the light rays emitted from a point \( x \) in a billiard table, we will see caustics too. The differential geometrical \( C_{x} \) will be dense however in general. In billiards, we have looked at the caustic of a family of rays which correspond to billiard trajectories on an invariant curve. However, there are some cases, where there is a direct connection between differential geometrical caustics and caustics of billiards. We can deform a sphere in such a way that the caustic of a point on the sphere is the caustic of a special billiard table. We have used this construction once to find metrics on spheres for which the caustics is nowhere differentiable.

CAUSTICS OF BILLIARDS. Caustics of billiards can be quite complicated. To the right, we see some examples for billiards in tables of equal thickness.