THE DYNAMICS OF 2-GENERATOR SUBGROUPS OF PSL(2, C)

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A classical result of Shimizu and Leutbecher (see, for instance [6], p. 59) asserts that if $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ generate a discrete subgroup of $PSL(2, \mathbf{C})$, then either $c = 0$ or $|c| \geq 1$. This has been strengthened by T. Jørgensen [4] as follows:

JØRGENSEN'S INEQUALITY. If X and Y generate a discrete, non-elementary subgroup of $PSL(2, \mathbf{C})$, then

$$|\mathrm{tr}^2(X) - 4| + |\mathrm{tr}(XYX^{-1}Y^{-1}) - 2| \geq 1 \ .$$

In this paper, we will show the existence of a sequence of inequalities, generalizing Jørgensen's inequality, which X and Y must satisfy in order for $\langle X, Y \rangle$, the group generated by X and Y, to be discrete. These conditions are mutually independent in the sense that, for given X and Y, at most one can fail to hold. These conditions arise from the Shimizu-Leutbecher process defined below.

For convenience, consider the upper half space model of hyperbolic 3-space. We denote a directed geodesic ℓ by the ordered pair of its end-points; so $\ell = (a, b)$, $a, b \in \mathbf{C}$, $a \neq b$. The *complex distance* $\tau = \delta(\ell_1, \ell_2)$

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between two directed geodesics $\ell_1 = (a_1, b_1)$ and $\ell_2 = (a_2, b_2)$ is defined as follows: $\tau \in \mathbb{C}$; $\mathrm{Re}(\tau) \geq 0$ is the hyperbolic distance between the geodesics; $\mathrm{Im}(\tau)$ is the angle made by the geodesics along their common perpendicular and is determined modulo 2π unless $\mathrm{Re}(\tau) = 0$, in which case $\pm \mathrm{Im}(\tau)$ is determined modulo 2π. One may compute the complex distance by the formula:

$$\cosh^2(\tau/2) = (a_1, a_2, b_2, b_1) ,$$

where (z_1, z_2, z_3, z_4) is the usual cross ratio, as can be checked if $\ell_1 = (-1, 1)$ and $\ell_2 = (-e^\tau, e^\tau)$.

Let X be a loxodromic element of $PSL(2, \mathbb{C})$ and axis(X) the directed geodesic in hyperbolic space joining the fixed points of X. If ℓ is a perpendicular to axis(X), then the complex distance τ between ℓ and $X(\ell)$ is called the *complex translation length* of X. In fact X translates $\mathrm{Re}(\tau)$ units along axis(X) and rotates hyperbolic space by $\mathrm{Im}(\tau)$ about axis(X). We have

$$\mathrm{tr}^2 X = 4 \cosh^2(\tau/2) ,$$

which makes sense even if X is not loxodromic.

Given X loxodromic with complex translation length τ, and Y in $PSL(2, \mathbb{C})$, one may check the formula:

$$\mathrm{tr}((YXY^{-1})X^{-1}) - 2 = -(1 - \cosh(\tau))(1 - \cosh(\beta)) ,$$

for β the complex distance from axis(X) to axis(YXY^{-1}); this follows by normalizing

$$X = \begin{pmatrix} \cosh(\tau/2) & \sinh(\tau/2) \\ \sinh(\tau/2) & \cosh(\tau/2) \end{pmatrix}$$

$$YXY^{-1} = \begin{pmatrix} \cosh(\tau/2) & e^{\beta}\sinh(\tau/2) \\ e^{-\beta}\sinh(\tau/2) & \cosh(\tau/2) \end{pmatrix}.$$

Given X and Y elements of PSL(2, C) with X loxodromic, we define the Shimizu-Leutbecher sequence inductively by:

$$Y_1 = YXY^{-1}, \quad Y_{i+1} = Y_i X Y_i^{-1}.$$

Let τ be the complex translation length of X, and let β_i be the complex distance between axis(X) and axis(Y_i). A necessary condition for the group generated by X and Y to be discrete is that the set $\{\cosh(\beta_i)\}$ should form a discrete subset of $\mathbf{C}$.

The following lemma allows one to compute $\cosh(\beta_i)$ inductively:

LEMMA. $\cosh(\beta_{i+1}) = (1-\cosh(\tau))\cosh^2(\beta_i) + \cosh(\tau)$.

This follows from the hyperbolic law of cosines: if ℓ_0, ℓ_1, ℓ_2 are given, the law of cosines gives a formula for $\omega = \delta(\ell_1, \ell_2)$ in terms of $\tau_1 = \delta(\ell_0, \ell_1)$, $\tau_2 = \delta(\ell_0, \ell_2)$ and a which is the complex distance from the perpendicular between ℓ_0 and ℓ_1 to the perpendicular between ℓ_0 and ℓ_2. The formula is:

$$\cosh(\omega) = \cosh(\tau_1)\cosh(\tau_2) - \cosh(a)\sinh(\tau_1)\sinh(\tau_2).$$

The lemma follows by setting $\tau_1 = \tau_2 = \beta_i$ and $a = \tau$. One way to check the law of cosines is to normalize so that $\ell_0 = (0, \infty)$, $\ell_1 = (t_1, t_1^{-1})$, and $\ell_2 = (et_2, et_2^{-1})$ where $t_1 = \tanh(\tau_1/2)$, $t_2 = \tanh(\tau_2/2)$, and $e = e^a$; then compute $\cosh^2(\omega/2) = (t_1, et_2, et_2^{-1}, t_1^{-1})$. Note that ℓ_2 does indeed have complex distance τ_2 to ℓ_0 with $(-e^a, e^a)$ as common perpendicular.

Now let $z_i = (1-\cosh(\tau))(\cosh(\beta_i))$. We may rewrite the above inductive formula as:

$$z_{i+1} = z_i^2 + C, \quad \text{where} \quad C = (1-\cosh(\tau))(\cosh \tau),$$

and we have that if X and Y generate a discrete group, then $\{z_i\}$ forms a discrete subset of $\mathbf{C}$.

The dynamical behavior of $\mathbf{C}$ under a quadratic polynomial is well understood from the work of Fatou-Julia ([1], [5]; see also [2]). Let

$f^i(z) = \underbrace{f \circ f \cdots \circ f}_{i \text{ times}}(z)$, where $f(z) = z^2 + C$; a solution ρ of the polynomial equation $f^i(z) = z$ will be called a stable periodic point of period i if $\left|\frac{d}{dz} f^i(\rho)\right| < 1$. Then f^i is contracting on any disk $B_\varepsilon = \{z : |z-\rho| < \varepsilon\}$ on which $\left|\frac{d}{dz} f^i\right| < 1$. The theorem of Fatou-Julia ensures that, for any choice of C, there is at most one stable periodic orbit. Further results of Fatou-Julia allow one to draw by computer the region E of $\mathbf{C}$ defined by $E = \{z : f^i(z)$ converges to the stable periodic orbit$\}$ (see Fig. 1), and the region of $\mathbf{C}$ defined by $\{C : z^2 + C$ has a stable periodic orbit$\}$ (see Fig. 2).

To obtain the above-mentioned inequalities, let p be a stable periodic point of f of period n; we may assume that $|p| < 1/2$. Expanding $f^n(z) = \sum_{i=0}^{2^n} a_i(z-p)^i$ as a Taylor series about p, we have

$$|f^n(z)-p| = |z-p|\left|\sum_{i=1}^{2^n} a_i(z-p)^{i-1}\right| \le |z-p|(2^n-1)m\,[\max(1, |z-p|^{2^n-1}]$$

where $m = \max(|a_i|)$. Setting $K < \frac{1 - \left|\frac{d}{dz} f^n(p)\right|}{(2^n-1)m}$, we see that on the disk $|z-p| < \min(K, 1)$, f^n is a contracting map. If also $K < \frac{\left|\frac{d}{dz} f^n(p)\right|}{m \cdot (2^n-1)}$, then $f^n(z) - p$ has no roots other than p in the disk $|z-p| < K$.

In the case $n = 1$, the fixed points of $f(z) = z^2 + (1-\cosh(\tau))(\cosh(\tau))$ are $\cosh(\tau)$, $1-\cosh(\tau)$. If $|1-\cosh(\tau)| < \frac{1}{2}$, we may set $p \le 1-\cosh(\tau)$, and $\frac{df}{dz}(p) = 2(1-\cosh(\tau))$, $m = 1$. We thus have the inequality: If $0 < |(1-\cosh(\tau))(\cosh(\beta)-1)| < \min(1-2|\cosh(\tau)-1|,\ 2|\cosh(\tau)-1|)$, then $\langle X, Y \rangle$ is not discrete. In view of our expressions for $\cosh(\tau)$ and $\cosh(\beta)$ given above, this becomes Jørgensen's inequality.

In the case $n = 2$, the periodic points of order 2 of $f(z) = z^2 + C$ are $\frac{-1 \pm \sqrt{1-4(C+1)}}{2}$, and $\frac{d}{dz} f^2(p) = 4(C+1)$. Using the estimates

$|p| < \frac{1}{2}$, $|C| < \frac{5}{4}$, we find $m \leq 4$, so that $0 < |z-p| < \frac{1}{12}\min(1-4|C+1|, 4|C+1|)$ implies $<X, Y>$ is not discrete.

In terms of the matrix functions of X and Y, this may be rewritten as:

$$0 < \left|\operatorname{tr}(XYX^{-1}Y^{-1}) - \frac{\operatorname{tr}^2(X)}{2} - \frac{-1 \pm \sqrt{(\operatorname{tr}^2(X) - 5(\operatorname{tr}^2(X)-1)}}{2}\right|$$

$$< \frac{1}{12}\min\left[(1 - |\operatorname{tr}^4(X) - 6\operatorname{tr}^2(X) - 4|, |\operatorname{tr}^4(X) - 6\operatorname{tr}^2(X) + 4|)\right]$$

implies $<X, Y>$ is not discrete.

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Fig. 1. The set E for $f(z) = z^2 + C$ with $C = .1 + .6i$.

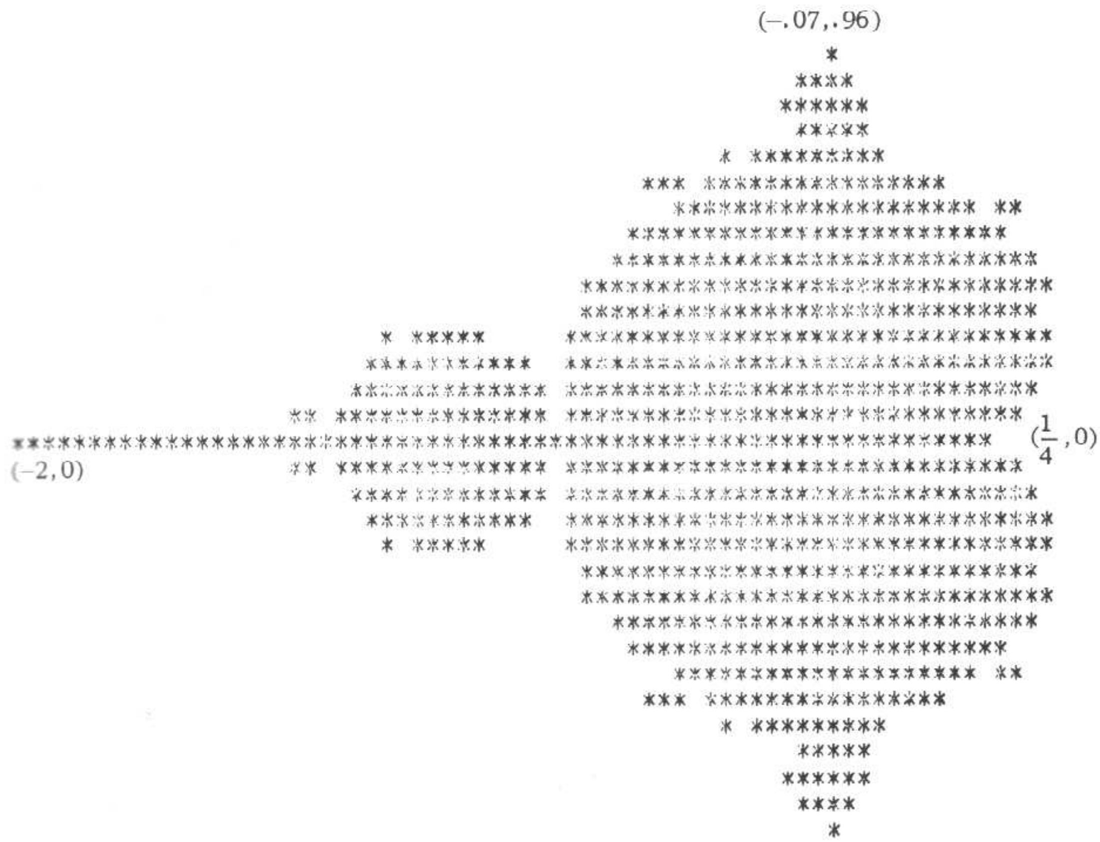

Fig. 2. The set of C's such that $f(z) = z^2 + C$ has a stable periodic orbit.

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