Derivation of the Poisson Kernel
From the Cauchy Formula

The Cauchy’s integral formula states that
\[
f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)d\zeta}{\zeta - z}
\]
for \(|z| < 1\) if \(f\) is holomorphic on the open unit 1-disk and continuous up to the boundary. We want to make the kernel real so that we can get a kernel for harmonic functions by taking the real part of the formula. Let \(\zeta = e^{i\theta}\). Then
\[
d\zeta = i e^{i\theta} d\theta = i \zeta d\theta
\]
and
\[
(\#) \quad f(z) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(\zeta) \frac{\zeta}{\zeta - z} d\theta.
\]
So we try to make \(\frac{\zeta}{\zeta - z}\) real. The easiest way is to take the complex conjugate
\[
\frac{\overline{\zeta}}{\overline{\zeta} - \overline{z}} = \frac{\zeta \overline{\zeta}}{\zeta \overline{\zeta} - \zeta \overline{z}} = \frac{1}{1 - \zeta \overline{z}} = \frac{\zeta}{\zeta(1 - \zeta \overline{z})}
\]
and
\[
\frac{\overline{\zeta}}{\overline{\zeta} - \overline{z}} d\theta = \frac{1}{i \zeta(1 - \zeta \overline{z})} d\theta = \frac{1}{i} \frac{d\zeta}{\zeta(1 - \zeta \overline{z})}.
\]
Hence
\[
\frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(\zeta) \frac{\overline{\zeta}}{\overline{\zeta} - \overline{z}} d\theta = \frac{1}{2\pi i} \int_{\theta=0}^{2\pi} f(\zeta) \frac{d\zeta}{\zeta(1 - \zeta \overline{z})} = f(0).
\]
Since
\[
f(0) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(\zeta) d\theta,
\]
it follows that
\[
\frac{1}{2\pi} \int_{\theta=0}^{2\pi} \left( \frac{\overline{\zeta}}{\overline{\zeta} - \overline{z}} - 1 \right) f(\zeta) = 0.
\]
We add this to (\#) and obtain
\[
f(z) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(\zeta) \left( \frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta} - \overline{z}} - 1 \right) d\theta
\]
We now rewrite the reproducing kernel as
\[
\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\zeta - \bar{z}} - 1 = 2 \text{Re} \left( \frac{\zeta}{\zeta - z} - \frac{1}{2} \right) = 2 \text{Re} \frac{2\zeta - \zeta + z}{2(\zeta - z)} = \text{Re} \frac{\zeta + z}{\zeta - z}.
\]

This is the Poisson kernel. We can also write
\[
\text{Re} \frac{\zeta + z}{\zeta - z} = \frac{\overline{\zeta} - z\z + z\overline{\zeta} - z\overline{z}}{|\zeta - z|^2} = \frac{\zeta + z\overline{z}}{|\zeta - z|^2}.
\]

The Poisson integral formula is
\[
f(z) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(\zeta) \text{Re} \frac{\zeta + z}{\zeta - z} \, d\theta
\]
or equivalently
\[
f(z) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(\zeta) \frac{\overline{\zeta} - z\overline{z}}{|\zeta - z|^2} \, d\theta.
\]