Homework on Partial Differential Equations
(No Need to Hand In)

Problem 1 (#2 on Page 256 of Strauss). Determine the vibrations
\[ u_{tt} = c^2 (u_{xx} + u_{yy}) \]
of a circular drumhead of radius \( a \) (held fixed on the boundary) with the
initial conditions
\[ u = 1 - \frac{r^2}{a^2} \text{ when } t = 0 \]
and \( u_t \equiv 0 \) when \( t = 0 \). (The answer is given on Page 410 of Strauss.)

Problem 2 (#4 on Page 263 of Strauss). Solve the wave equation
\[ u_{tt} - c^2 \Delta u = 0 \]
on the ball \( \{ r < a \} \) of radius \( a \) in \( \mathbb{R}^3 \) with the conditions
\[ \frac{\partial u}{\partial r} = 0 \text{ on } \{ r = a \}, \]
\[ u = z = r \cos \theta \text{ when } t = 0, \]
\[ u_t \equiv 0 \text{ when } t = 0, \]
where \( \theta \) is the colatitude and \( \varphi \) is the longitude in the spherical coordinates
\((r, \theta, \varphi)\) of \( \mathbb{R}^3 \) so that \( x = r \sin \theta \cos \varphi, y = r \sin \theta \sin \varphi, \) and \( z = r \cos \theta \). (The
answer is given on Page 410 of Strauss.)

Problem 3 (#18 on Page 274 of Strauss). Find an equation for the eigen-
values \( \lambda \) and find the eigenfunctions \( v \) of the negative Laplacian \( -\Delta \) in the
disk \( \{ x^2 + y^2 < a^2 \} \) for eigenvalues \( \lambda \) (in the sense that \( -\Delta v = \lambda v \)) with the
boundary condition
\[ \frac{\partial v}{\partial r} + hv = 0 \]
on the circle \( \{ x^2 + y^2 = a^2 \} \), where \( h \) is a constant. (The answer is given on
Page 411 of Strauss.)
Problem 4 (#1 and #4 on Page 278 of Strauss). Consider the \( \ell \)-th Legendre polynomial
\[
P_\ell(z) = \frac{1}{2^\ell} \sum_{j=0}^{m} \frac{(-1)^j}{j!} \frac{(2\ell - 2j)!}{(\ell - 2j)!((\ell - j)!)} z^{\ell-2j},
\]
where \( m = \frac{\ell}{2} \) if \( \ell \) is even, and \( m = \frac{\ell-1}{2} \) if \( \ell \) is odd.

(a) Prove the following recurrent relation for Legendre polynomials
\[
(\ell + 1)P_{\ell+1}(z) - (2\ell + 1)zP_\ell(z) + \ell P_{\ell-1}(z) = 0.
\]

(b) Show that
\[
\int_{x=-1}^{1} x^2 P_\ell(x) \, dx = 0
\]
for \( \ell \geq 3 \).

Problem 5 (#5 on Page 278 of Strauss). Let \( f(x) = x \) for \( 0 \leq x < 1 \), and \( f(x) = 0 \) for \( -1 < x \leq 0 \). Find the coefficients \( a_\ell \) in the expansion
\[
f(x) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(x)
\]
of \( f(x) \) in the interval \((-1, 1)\) in terms of Legendre polynomials
\[
P_\ell(z) = \frac{1}{2^\ell} \sum_{j=0}^{m} \frac{(-1)^j}{j!} \frac{(2\ell - 2j)!}{(\ell - 2j)!((\ell - j)!)} z^{\ell-2j},
\]
where \( m = \frac{\ell}{2} \) if \( \ell \) is even, and \( m = \frac{\ell-1}{2} \) if \( \ell \) is odd. (The answer is given on Page 411 of Strauss.)

Problem 6 (#7 on Page 278 of Strauss). Find the harmonic function in the ball \( \{ x^2 + y^2 + z^2 < a^2 \} \) with the boundary condition \( u = A \) on the top hemisphere \( \{ x^2 + y^2 + z^2 = a^2, \ z > 0 \} \) and with \( u = B \) on the bottom hemisphere \( \{ x^2 + y^2 + z^2 = a^2, \ z < 0 \} \), where \( A \) and \( B \) are constants. (The answer is given on Page 411 of Strauss.)